Math 231
1/4/11

Feb 16 Midterm 1, 7.1 – 7.8 (w/o 7.6)
8.1 – 8.2

(20)

Approximate Integration.

We cannot always explicitly compute a definite integral. In this case there are many methods for approximating the integral, and moreover, we can determine how good the estimate is.

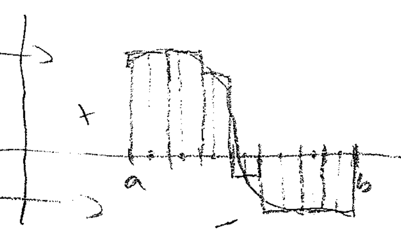
All the methods we will discuss are based on approximating the function using geometric techniques.

Recall: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

where n is an integer, $\Delta x = \frac{b-a}{n}$ & x_i^* is any point in the i^{th} interval $a+(i-1)\Delta x \le x_i^* \le a+i\Delta x$.

the sum represents the signed area

since $f(x_i^*)\Delta x = \begin{cases} \text{area of box} & \text{if } f(x_i^*)>0 \\ -(\text{area of box}) & \text{if } f(x_i^*)<0 \end{cases}$

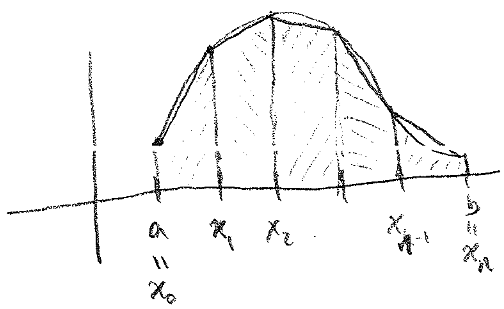


Choosing the midpoint, $x_i^* = a+(i-\frac{1}{2})\Delta x$, of the i^{th} interval gives the midpoint rule:

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \Delta x \sum_{i=1}^{n} f(x_i^*), \quad \Delta x = \frac{b-a}{n}$$

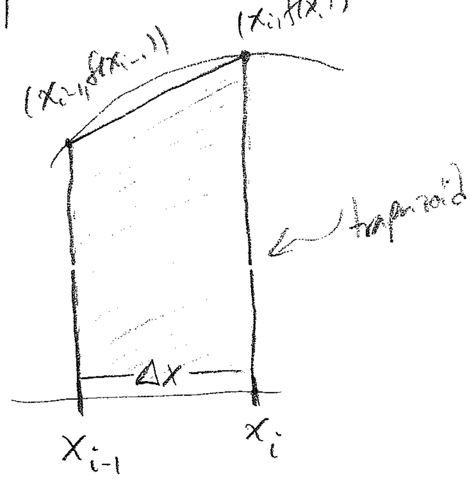
$x_i^* = a+(i-\frac{1}{2})\Delta x, \quad i=1,\dots,n.$

We can also approximate using of shapes:



$\Delta x = \frac{b-a}{n}$

$x_i = a + i\Delta x$



Area of trapezoid $= \Delta x\left(\frac{1}{2}(f(x_{i-1}) + f(x_i))\right)$

Trapezoidal rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\sum_{i=1}^{n}\left(f(x_{i-1}) + f(x_i)\right) = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

EX Estimate with $n=4$, $\int_0^2 1+x^3\,dx$

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

Midpoint: $x_i^* = 0 + (i - \frac{1}{2})\frac{1}{2} = \frac{2i-1}{4}$

$$M_4 = \frac{1}{2}\left(1 + (\tfrac{1}{4})^3 + 1 + (\tfrac{3}{4})^3 + 1 + (\tfrac{5}{4})^3 + 1 + (\tfrac{7}{4})^3\right)$$

$$= \frac{4}{2} + \frac{1}{128}(1 + 27 + 125 + 343) = 2 + \frac{496}{128} = 5.875$$

Trapezoidal: $x_i = 0 + i\Delta x = \frac{i}{2}$

$$T_n = \frac{1/2}{2}\left(1 + 0^3 + 2(1 + (1/2)^3) + 2(1 + 1^3) + 2(1 + (3/2)^3) + 1 + 2^3\right)$$

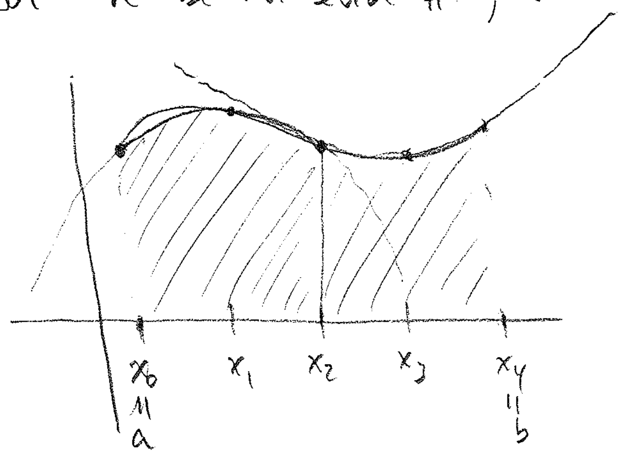
$$= \frac{1}{4}\left(8 + 1/4 + 2 + \frac{27}{4} + 8\right) = \frac{1}{4}(25) = \frac{25}{4} = 6.25$$

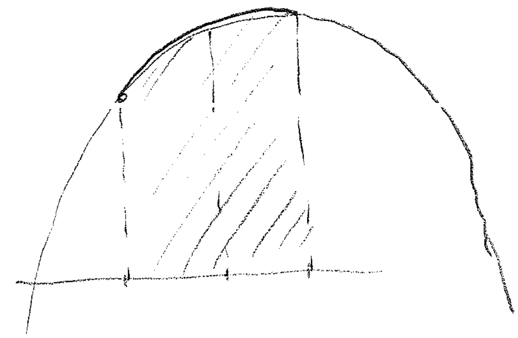
Actual: $x + \frac{x^4}{4}\Big|_0^2 = 2 + 4 - 0 = 6$

Another method approximates the integral by approximating the function on each interval by a parabola — Simpson's rule.

To approximate $\int_a^b f(x)\,dx$,

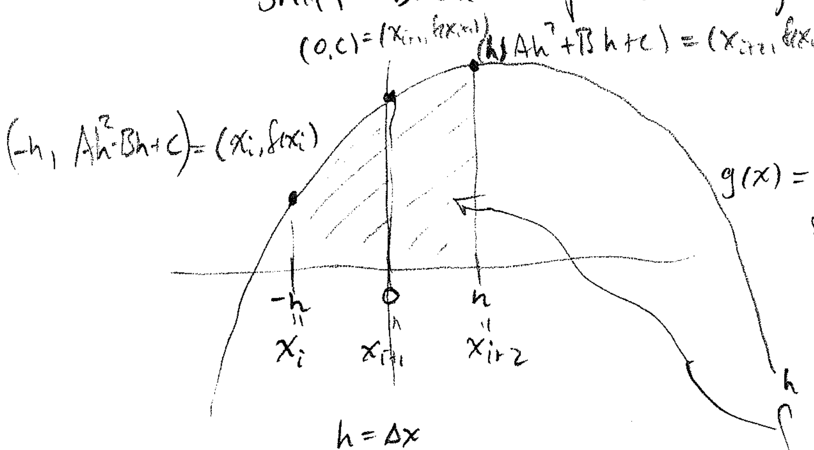
Let n be an even #, $\Delta x = \frac{b-a}{n}$





Need to compute area under a parabola on 3 equally spaced points

Shift back so points are symmetric about 0.



$$\Rightarrow \quad \begin{aligned} f(x_i) &= Ah^2 - Bh + C \\ f(x_{i+1}) &= C \\ f(x_{i+2}) &= Ah^2 + Bh + C \end{aligned}$$

$$\Rightarrow f(x_i) + f(x_{i+2}) = 2Ah^2 + 2C$$

$$\int_{-h}^{h} Ax^2 + Bx + C\,dx = \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \Big|_{-h}^{h}$$

$$= \frac{Ah^3}{3} + Ch + \frac{Ah^3}{3} + Ch$$

$$= 2h\left(\frac{Ah^2}{3} + C\right)$$

$$= \frac{h}{3}(2Ah^2 + 6C)$$

$$= \frac{h}{3}\left(f(x_i) + f(x_{i+2}) + 4f(x_{i+1})\right)$$

$$= \frac{\Delta x}{3}\left(f(x_i) + 4f(x_{i+1}) + f(x_{i+2})\right)$$

So, from we get Simpson's rule

$$\int f(x)\,dx \approx \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + f(x_2)$$
$$+ f(x_2) + 4f(x_3) + f(x_4)$$
$$+ \dots$$
$$+ f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \Big)$$

$$= \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

$$= S_n$$

Ex Estimate with $n=4$ $\int_0^2 1+x^3\,dx$

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$\int 1+x^3\,dx \approx \frac{1}{6}\left(1 + 4\left(1+\left(\tfrac{1}{2}\right)^3\right) + 2\left(1+1^3\right) + 4\left(1+\left(\tfrac{3}{2}\right)^3\right) + 1+2^3\right)$$

$$= \frac{1}{6}\left(22 + \frac{1}{2} + \frac{27}{2}\right) = \frac{1}{6}\left(22+14\right) = \frac{36}{6} = 6$$

Get exact value (we'll see why shortly), but in any case get better estimates in general.

Error bounds Let f be a function on $[a,b]$, n an integer

If E_M, E_T, and E_S are the errors from each of the rules:

$$\int_a^b f(x)dx - M_n = E_M, \quad \int_a^b f(x)dx - T_n = E_T, \quad \int_a^b f(x)dx - S_n = E_S$$

(we assume n is even for this last)

then $$|E_M| \le \frac{K(b-a)^3}{24n^2}, \quad |E_T| \le \frac{K(b-a)^3}{12n^2}$$

and $$|E_S| \le \frac{K'(b-a)^5}{180\,n^4}$$ where $K = \max\limits_{a\le x\le b} |f''(x)|$

$$K' = \max_{a\le x\le b} |f^{(4)}(x)|$$

In our examples, we could compute $\int_a^b f(x)dx$. In general, we can't, so we want to know how good our estimate is — how much can we trust it. These bounds provide such estimates.

Ex How large should n for each of the 3 rules to guarantee an approx. of $\int_0^1 e^{x^2}dx$ within $.000001 = \frac{1}{1000000}$?

$f(x) = e^{x^2}$

$f'(x) = 2xe^{x^2}$

$f''(x) = ((2x)^2 + 2)e^{x^2} = (4x^2+2)e^{x^2}$

$f'''(x) = (8x + (4x^2+2)2x)\, e^{x^2} = (8x^3 + 12x)\, e^{x^2}$

$f^{(4)}(x) = (24x^2 + 12 + 2x(8x^3 + 12x))\, e^{x^2} = (16x^4 + 48x + 12)\, e^{x^2}$

$K = \max\limits_{0 \le x \le 1} |f''(x)| \le f''(1) = 6e \le 16.4$

f'' is increasing.

$K' = \max\limits_{0 \le x \le 1} |f^{(4)}(x)| \le f^{(4)}(1) = 76e \le 206.6$

$f^{(4)}$ is increasing

Choose n so that

$$\frac{K(1-0)^3}{24n^2} \le \frac{16.4}{24n^2} \le \frac{1}{1{,}000{,}000} \qquad n^2 \ge 683333.\overline{3} \quad \text{so } n \ge 827$$

$$\frac{K}{12n^2} \le \frac{16.4}{12n^2} \le \frac{1}{1{,}000{,}000} \qquad n^2 \ge 1366666.\overline{6} \quad \text{so } n \ge 1170$$

$$\frac{K'}{180n^4} \le \frac{206.6}{180n^4} \le \frac{1}{1000000} \qquad n^4 \ge 1147778 \quad \text{so } n \ge 33.$$