algebra $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ so

$$S_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

so $\lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$ and hence

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Results not typical – can't usually find sum
just decide convergence/divergence

EX Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} = ?$$

just look at part of this.

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2}$

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$\vdots$

$S_{2^n} \geq 1 + n\left(\frac{1}{2}\right)$

since $\lim_{n\to\infty} S_{2^n} = \infty$ and $S_1 < S_2 < S_3 < \Rightarrow \lim_{n\to\infty} S_n = \infty$ ✓

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observe if $\sum_{n=1}^{\infty} a_n$ converges, so that partial sums converge

$s = \lim_{n\to\infty} S_n$, we also have $\lim_{n\to\infty} S_{n-1} = S$ and so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

So, if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

What's going on? $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\lim_{n\to\infty} \frac{1}{n} = 0$ ... ??

Look at the statement again:

IF $\sum_{n=1}^{\infty} a_n$ converges THEN $\lim_{n\to\infty} a_n = 0$.

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we cannot draw any conclusion from this statement (though we do know $\lim_{n\to\infty} \frac{1}{n} = 0$ — this is irrelevant)

Observe however:

IF $\lim_{n\to\infty} a_n \neq 0$ (or it diverges) THEN $\sum_{n=1}^{\infty} a_n$ diverges.

[Why? well, if $\sum_{n=1}^{\infty} a_n$ did not diverge, then $\lim_{n\to\infty} a_n = 0$, which is false]

WARNING $\lim_{n\to\infty} a_n$ can only be used to prove divergence.

DO NOT say: $\lim_{n\to\infty} a_n = 0$ so $\sum_{n=1}^{\infty} a_n$ converges.

Convergent series obey some arithmetic rules

Theorem If c is a real #, $\sum a_n$, $\sum b_n$ convergent, then

- $\sum (a_n + b_n) = \sum a_n + \sum b_n$ convergent.
- $\sum c a_n = c \sum a_n$
- $\sum (a_n - b_n) = \sum a_n - \sum b_n$

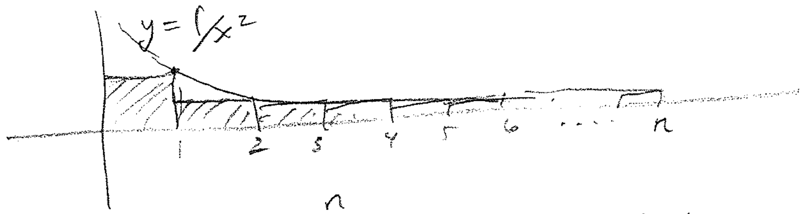
follows from related facts for sequences.

Tests for convergence/divergence

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Argue geometrically that this converges:

$S_n = \sum_{i=1}^{n} \frac{1}{i^2}$ = sum of areas of n rectangles below:



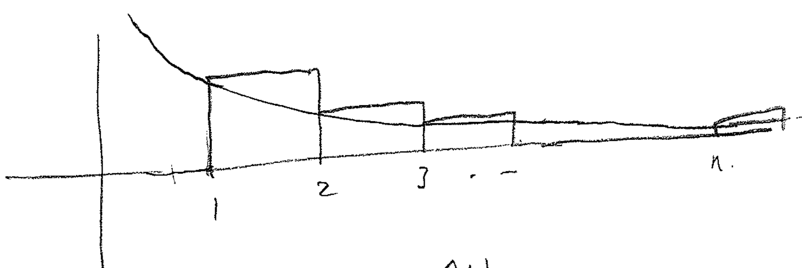
$$S_n \le 1 + \int_1^n \frac{dx}{x^2} = 1 + \left(-\frac{1}{x}\right)\Big|_1^n = 2 - \frac{1}{n} \le 2.$$

$$S_{n+1} = S_n + \frac{1}{(n+1)^2} > S_n$$

So $\{S_n\}_{n=1}^{\infty}$ is increasing & bounded, hence convergent.

Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

We can make a similar argument to show $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent:



$$S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \ge \int_1^{n+1} \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_1^{n+1} = 2\sqrt{n+1} - 2$$

So, $\lim_{n\to\infty} S_n = \infty$ and S_n is divergent, so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

These are special cases of

Integral Test Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and $a_n = f(n)$. Then:

① $\sum_{n=1}^{\infty} a_n$ is convergent if $\int_1^{\infty} f(x)\,dx$ is convergent

② $\sum_{n=1}^{\infty} a_n$ is divergent if $\int_1^{\infty} f(x)\,dx$ is divergent

That is $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

[Either both converge or both diverge].

Same reasoning: S_n is increasing and

$$S_n \le a_1 + \int_1^n f(x)\,dx \le a_1 + \int_1^{\infty} f(x)\,dx$$

and

$$S_n \ge \int_1^{n+1} f(x)\,dx$$

If $\int_1^{\infty} f(x)\,dx$ converges, then S_n is bounded & increasing, hence convergent.

If $\int_1^{\infty} f(x)\,dx$ diverges, then $\left\{\int_1^{n+1} f(x)\,dx\right\}_{n=1}^{\infty}$ diverges

[because ⓐ $\lim_{t\to\infty} \int_1^t f(x)\,dx$ diverges and ⓑ $\int_1^t f(x)\,dx$ is increasing in t, since $f(x) > 0$.]

Ex

p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$

since same statement holds for $\int_1^{\infty} \frac{dx}{x^p}$.