Math 231
2/25/11

Series

An expression of the form

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad a_1 + a_2 + a_3 + \dots + a_n + \dots \qquad (a_n = n^{th} \text{ term of series})$$

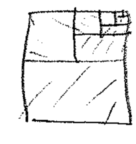
where $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers is called a (real) infinite series or just series.

Note: addition is only defined for a finite set of numbers, so what does this mean? Is this a number?

Answer: some times...

$$1+1+1+1+1+1+\dots = \sum_{n=1}^{\infty} 1 \quad \text{should not be a number.}$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\dots = \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{should be } 1:$$



What about

$$-1+1+(-1)+1+(-1)+1+\dots = \sum_{n=1}^{\infty} (-1)^n \quad ??$$

$$-1 + \underbrace{1+(-1)}_{0} + \underbrace{1+(-1)}_{0} + \underbrace{1+(-1)}_{0} + \dots = -1$$

or

$$\underbrace{-1+1}_{0} + \underbrace{(-1)+1}_{0} + \underbrace{(-1)+1}_{0} + (-1)+1 + \dots = 0$$

So, we need to take some care in deciding whether or not $\sum_{n=1}^{\infty} a_n$ should be a number, and if so, what it should be.

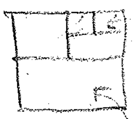
Defn If $\sum_{n=1}^{\infty} a_n$ is a series, the n^{th} partial sum $s_n = \sum_{i=1}^{n} a_i$.

Say series converges if $\{s_n\}_{n=1}^{\infty}$ converges, and sum of series is $\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n}$ in this case.

Ex How about $\sum_{n=1}^{\infty} \frac{1}{2^n}$? Do we get 1 ?

$$S_n = \sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} \underset{\substack{\uparrow \\ \text{alg. trick}}}{=} \frac{1-\frac{1}{2}}{1-\frac{1}{2}}\left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}\right)$$

geometrically



$$= \frac{1}{1/2}\left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} - \left(\frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n+1}}\right)\right)$$

$$= 2\left(\frac{1}{2} - \frac{1}{2^{n+1}}\right) = 1 - \frac{1}{2^n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1. \checkmark$$

this is a special case of <u>geometric series</u> :

$$\sum_{n=1}^{\infty} ar^{n-1} \qquad \left(\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} \right)$$

same trick :

$$S_n = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$$

$$= a\left(1 + r + r^2 + \dots + r^{n-1}\right)\left(\frac{1-r}{1-r}\right) \qquad \underline{r \neq 1.}$$

$$= \frac{a}{1-r}\left(1 + r + r^2 + \dots + r^{n-1} - (r + r^2 + \dots + r^n)\right)$$

$$= \frac{a}{1-r}\left(1 - r^n\right)$$

$$\lim_{n\to\infty} \frac{a}{1-r}(1-r^n) = \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n$$ converges iff $-1 < r < 1$

in this case $\lim_{n\to\infty} r^n = 0$ so

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ if } |r| < 1 \text{ and diverges if } |r| \geq 1}$$

Ex $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$S_n = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots + \frac{1}{n(n+1)} \;?$$

algebra $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ so

$$S_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

so $\lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$ and hence

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Results not typical - can't usually find sum just decide convergence/divergence

<u>EX</u> Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u>

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} = ?$$

just look at part of this.

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2}$

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$\vdots$

$S_{2^n} \geq 1 + n\left(\frac{1}{2}\right)$

since $\lim_{n\to\infty} S_{2^n} = \infty$ and $S_1 < S_2 < S_3 < \Rightarrow \lim_{n\to\infty} S_n = \infty$ ✓

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observe if $\sum_{n=1}^{\infty} a_n$ converges, so that partial sums converge

$S = \lim_{n\to\infty} S_n$, we also have $\lim_{n\to\infty} S_{n-1} = S$ and so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

$$\boxed{\text{So, if } \sum_{n=1}^{\infty} a_n \text{ converges, then } \lim_{n\to\infty} a_n = 0}$$