Math 231
2-23-11

Theorem If f is a function, $\lim_{x\to\infty} f(x) = L$, and $a_n = f(n)$, then $\lim_{n\to\infty} a_n = L$.

What's the difference? For $\varepsilon > 0$, there is an M so that if $x \geq M$, then $|f(x) - L| < \varepsilon$. Let $N \geq M$ be an integer. then for $n \geq N$ an integer, $n \geq N \geq M$ and

$$|a_n - L| = |f(n) - L| < \varepsilon.$$

For $\lim_{x\to\infty} f(x) = L$, we need $|f(x) - L| < \varepsilon$ for all real numbers $\geq M$

For $\lim_{n\to\infty} a_n = L$, - - - - $|a_n - L| < \varepsilon$ for all integers $\geq N$.

The converse is false:

EX $f(x) = \cos(2\pi x)$. Then $a_n = f(n) = 1$ for every n.

so $\lim_{n\to\infty} a_n = 1$, but $\lim_{x\to\infty} f(x)$ does not exist.

— useful since can evaluate $\lim_{x\to\infty} f(x)$ w/ tools already know.

Defn $\lim_{n\to\infty} a_n = \infty$ means for every M, there is an N so that if $n \geq N$, then $a_n > M$.

Analogues of limit laws for functions also hold;

- sums, differences, products, quotients, powers —] see p678.
- squeeze theorem —

  If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$, and $a_n \leq c_n \leq b_n$, then $\lim_{n\to\infty} c_n = L$.
- If f is continuous and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Ex $\displaystyle\lim_{n\to\infty} \frac{3n^2-2n+1}{n^4+2n-1}$

$$\lim_{n\to\infty} \frac{3n^2-2n+1}{n^4+2n-1} = \lim_{n\to\infty} \frac{\overset{\to 0}{3/n^2} - \overset{\to 0}{2/n^3} + \overset{\to 0}{1/n^4}}{\underset{\downarrow 1}{1} + \underset{\downarrow 0}{2/n^2} - \underset{\downarrow 0}{1/n^4}} = \frac{\lim 3/n^2 - \lim 2/n^3 + \lim 1/n^4}{\lim 1 - \lim 2/n^2 - \lim 1/n^4} = \frac{0}{1} = 0$$

Ex

$$\lim_{n\to\infty} \frac{e^{1/n^2}}{1+1/n} = \lim_{n\to\infty} f(1/n) = f(0) = \frac{e^0}{1+0} = 1.$$

where $f(x) = \frac{e^{x^2}}{1+x}$ continuous at 0 and $\lim_{n\to\infty} \frac{1}{n} = 0$.

Ex $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n}$

$$0 \le \frac{n!}{n^n} = \frac{1\cdot 2 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\cdots\left(\frac{n}{n}\right) \le \frac{1}{n}$$

$\frac{j}{n} \le 1$ for every $j = 1, \dots, n$

squeeze theorem implies

$\displaystyle\lim_{n\to\infty} \frac{n!}{n^n} = 0$ since $\displaystyle\lim_{n\to\infty} 0 = 0$ $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$.

Ex r a real #: $\displaystyle\lim_{n\to\infty} r^n = ?$

$r=1$: $\displaystyle\lim_{n\to\infty} 1^n = 1$

$r=-1$: $\displaystyle\lim_{n\to\infty} (-1)^n$ diverges $-1, 1, -1, 1, -1, 1, -1, 1, -1, \dots$

$0<r<1$: think of r as a proportion less than 1.

$r^n = r(r^{n-1})$ — so r^n is a proportion less than 1 of r^{n-1}.

so $r^{n-1} \rightsquigarrow r^n$ shrinks a definite amount

therefore $\displaystyle\lim_{n\to\infty} r^n = 0$

Observe that if $-1<r<0$, then $0<|r|<1$ so

$-|r|^n \le r^n \le |r|^n$ squeeze theorem $\Rightarrow \lim_{n\to\infty} r^n = 0$

In general. $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

So, also have $|r|>1$, $\lim_{n\to\infty} r^n$ diverges. (since $(\frac{1}{r})^n \to 0$)

$$\boxed{\lim_{n\to\infty} r^n = \begin{cases} 1 & \text{if } r=1 \\ 0 & \text{if } -1<r<1 \\ \text{diverge} & \text{otherwise} \end{cases}}$$

increasing (nondecreasing):

$a_1 \le a_2 \le a_3 \le \ldots$

decreasing (nonincreasing)

$a_1 \ge a_2 \ge a_3 \ge \ldots$

If $\{a_n\}_{n=1}^{\infty}$ is one of these it's called <u>monotone</u>.

$\{a_n\}_{n=1}^{\infty}$ is bounded if there is a number M s.t. $|a_n| \le M$ for all n.

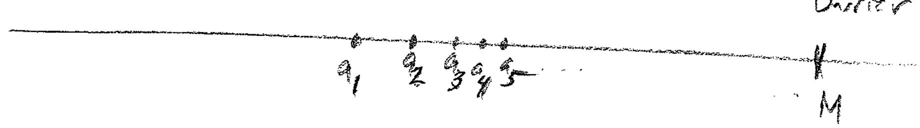
(bounded above/below $a_n \le M$, $a_n \ge m$.)

<u>Thm</u> Bounded monotone sequences converge.

Intuitively clear:

suppose $a_1 \le a_2 \le \ldots$ and $a_n \le M$ for all n.

barrier — a_n can't go past.



push barrier down as far as possible —

Smallest barrier L is "least upper bound

this exists — part of structure of real #'s —

observe that $L - a_n \geq 0$ and if $\lim_{n\to\infty} L - a_n \neq 0$, then $L - a_n > m > 0$

then $L - m$ is also a barrier, contradicts that th L was smallest.