Sequences and series

New direction & ideas. [start w/ notion of convergence of infinite sequences of real numbers, then step up to convergence of sums of an infinite sequence of real numbers, and finally convergence of sums of an infinite sequence of functions]

- infinite sequences of real numbers
- infinite sums of real numbers
- infinite sums of functions.

Defn A sequence of real numbers is an ordered list of an infinite set of numbers.

Ex

$\{1, 2, 3, 4, 5, 6, \ldots\}$

$\{1, 1/2, 1/4, 1/8, 1/16, 1/32, \ldots\}$

$\{0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots\}$

$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$

$\{\sin(1), \sin(2), \sin(3), \sin(4), \sin(5), \ldots\}$

1^{st} number in the sequence is called the 1^{st} term

2^{nd} " — — — — — — 2^{nd} term

⋮

n^{th} " — — — — — — n^{th} term

Often use subscripts as a way of keeping track of a sequence. —

$\{a_1, a_2, a_3, a_4, \ldots\}$

represents a sequence. The 1st term is a_1, 2nd is a_2, ... n^{th} term is a_n.

Also write $\{a_n\}_{n=1}^{\infty}$ or some times just $\{a_n\}$.

observe that we can think of a sequence as a function f with domain $\mathbb{N}$ = natural numbers = positive integers where

$f(n) = a_n$. — for every $n \in \mathbb{N}$, $a_n \in \mathbb{R}$ ...

convenient way to think about sequences, even if we don't use this notation

Sometimes use other indices, e.g. non-negative integers, integers at least 2 or 3, or 100 ... we write

$\{a_n\}_{n=0}^{\infty}$ or $\{a_n\}_{n=2}^{\infty}$ or $\{a_n\}_{n=100}^{\infty}$.

Ex

What are the rest?

$\{n\}_{n=1}^{\infty}$

$\{1/2^n\}_{n=0}^{\infty}$

$\{\frac{1}{2}(1+(-1)^n)\}_{n=1}^{\infty}$

have nice expressions for n^{th} term

$\{a_n\}_{n=1}^{\infty}$ defined recursively $a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$.

Fibonacci sequence

$\{\sin(n)\}_{n=1}^{\infty}$.

Our main consideration of sequences is <u>convergence</u>.

Informally, $\{a_n\}_{n=1}^{\infty}$ <u>converges to a number L</u> if the numbers a_n get closer and closer to L as n gets larger and larger. So, a_n's are approximating L better and better. How do we make this precise?

[Have students work on this]

<u>Def'n</u> $\{a_n\}_{n=1}^{\infty}$ converges to L, written $\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there is a natural number N so that $|a_n - L| < \varepsilon$ for every $n \geq N$.

<u>Ex</u> $\lim_{n\to\infty} \frac{1}{n} = 0$

For $\varepsilon > 0$, how large does n have to be so that
$|\frac{1}{n} - 0| < \varepsilon$?

This is the same as $\frac{1}{n} < \varepsilon$ or $n > \varepsilon$.
So, let N be an integer, $N > \varepsilon$, then $n \geq N$, $|\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon$.

<u>Ex</u> [student] Show $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

For $\varepsilon > 0$, how large does n have to be so that
$|\frac{n}{n+1} - 1| < \varepsilon$?

Want $\left|\frac{n-(n+1)}{n+1}\right| = \frac{1}{n+1} < \varepsilon$, so need $n+1 > \frac{1}{\varepsilon}$. Let $N > \frac{1}{\varepsilon} - 1$ be an integer, then $n \geq N > \frac{1}{\varepsilon} - 1$