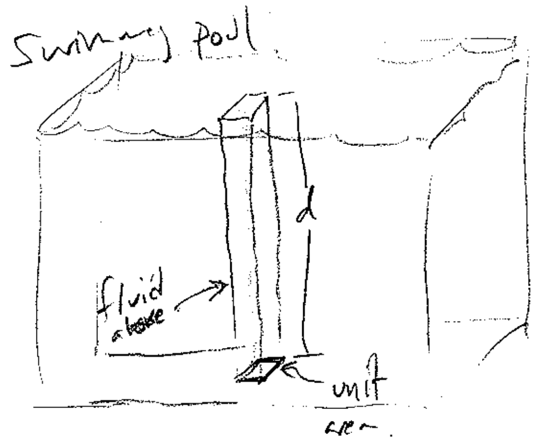
Applications to physics & engineering

— these could have been covered last semester; no new ideas.

Hydrostatic force & pressure

When you swim to the bottom of a deep pool, you feel the pressure [on your ears and nose it's also harder to breathe b/c of pressure on your chest]. This is because of the weight of the water above you, and the fact that the pressure is the same in all directions. Pressure is force/area, $P = F/A$, so



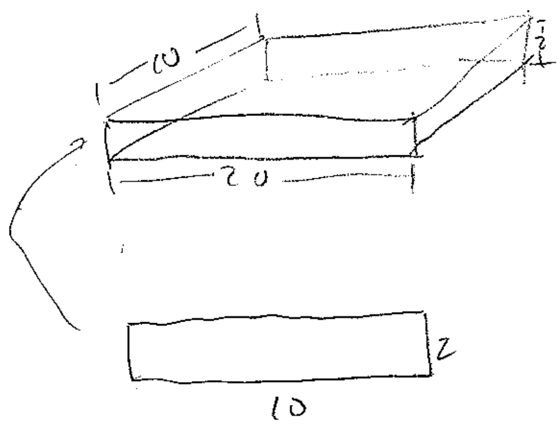
$$P = \frac{F}{A} = \frac{mg}{A} \qquad m = \text{mass},\ g = \text{acc. of gravity},\ A = \text{area}.$$

$$= \rho \frac{Vg}{A} \qquad \rho = \text{density},\ V = \text{volume}$$

$$= \rho g d. \qquad V = A \cdot d \quad d = \text{depth}$$

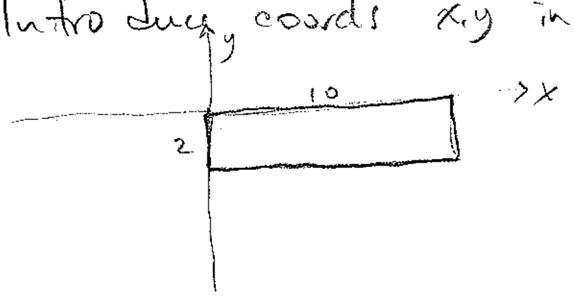
Hydrostatic Pressure is same in all directions, so e.g., pressure on walls of swimming pool filled with water also $\rho g d$ at depth d.

<u>EX</u> How much <u>hydrostatic force</u> is exerted on the short end of a 10x20 m pool, that is 2 m deep?



Can't just multiply area 2x10 times pressure b/c pressure is higher at bottom than at top

Introduce coords x, y in meters.



$-y$ = depth in meters

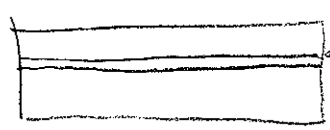
ρ = density of water = $1000 \, kg/m^3$

g = acc. of gravity = $9.8 \, m/s^2$.

Pressure on wall at point (x, y) is

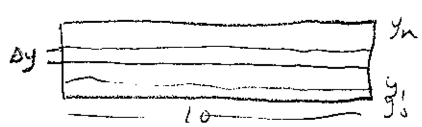
$$P(y) = -y\rho g = -9800y \quad N/m^2 = Pa \text{ (Pascals.)}$$

depends only on depth, so approximately the same on a short strip:



pressure ≈ constant

Can approximate total force as sum of forces on decomposition into strips.



Fix n = # of strips

$\Delta y = \frac{2}{n}$. $\quad y_i = -2 + i\Delta y$

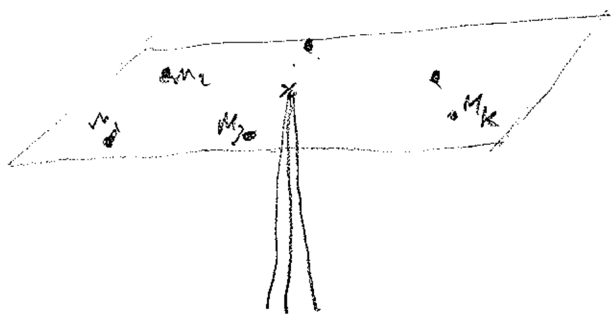
then $$F \approx \sum_{i=1}^{n} (\text{Area of strip}) \cdot P(y_i) = \sum_{i=1}^{n} 10 \cdot \Delta y \cdot (-9800y) \text{ kg}.$$

taking a limit, we get exact value

$$F = \int_{-2}^{0} -98000y \, dy = -\frac{98000y^2}{2}\Big|_{-2}^{0} = 49000(4) = 196000$$

Center of mass

Given a collection of point masses on the plane, say $m_1, \ldots, m_k$ at k points $(x_1, y_1), \ldots, (x_k, y_k)$, the center of mass is the balance point of the plane, physically:



$(\bar{x}, \bar{y})$ = balance point is called the center of mass and is computed as the weighted average of coordinates:

$$\bar{x} = \frac{\sum_{i=1}^{k} m_i x_i}{\sum_{i=1}^{k} m_i} \qquad \bar{y} = \frac{\sum_{i=1}^{k} m_i y_i}{\sum_{i=1}^{k} m_i}$$

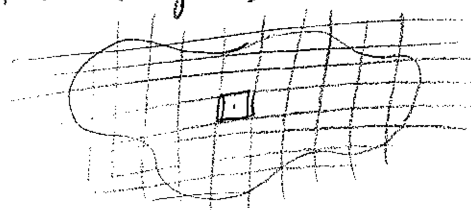
Based on Archimedes law of the lever, also intuitively clear: unchanged by scaling all masses, translates and rotates with a translation or rotation of pts, scales with scaling of points.

The numbers $\sum_1^k m_i x_i = M_y$, $\sum_1^k m_i y_i = M_x$ are called the moments w.r.t. y-axis / x-axis, respectively - tendency to rotate about those axes - if both zero, then balance at origin.

setting $m = \sum_1^k m_i$, total mass, then $\bar{x} = M_y/m$, $\bar{y} = M_x/m$

Can use this together with calculus to compute center of mass or <u>centroid</u> of a flat plate of uniform density ρ (= mass/Area)

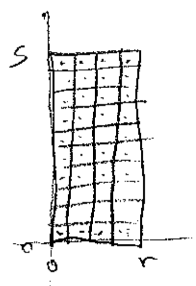
idea: cut up into small squares with area A.



observe: cut R into two pieces, find center of mass of each C_1, C_2, w/ masses m_1, m_2, then center of mass of $R = \frac{m_1C_1 + m_2C_2}{m_1+m_2}$.

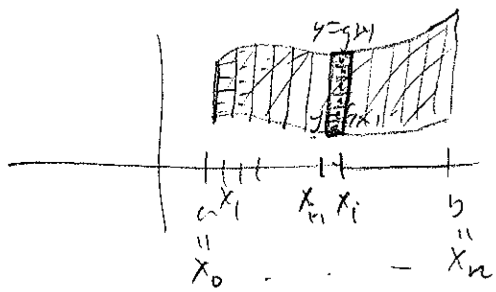
replace each square with point mass at center (or anywhere in square) w/ mass ρA, take a limit as size of squares $\to 0$.

Ex rectangular plate: centroid is center of rectangle —



matches intuition of symmetry principle: invariant by symmetry of plate

Ex plate = region between graphs $y=f(x)$, $y=g(x)$ on $[a,b]$ w/ $f(x) \le g(x)$



can cut up into rectangles, compute centers of mass, and mass, take weighted average:

$\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$,

$x_i^* = x_{i-1} + \frac{\Delta x}{2}$

$$\bar{x} \approx \frac{\displaystyle\sum_{i=1}^{n} \overbrace{\overbrace{\rho \Delta x \left(g(x_i) - f(x_i)\right)}^{\text{mass}} \, x_i^*}^{\text{moment wrt } y}}{\displaystyle\sum_{i=1}^{n} \rho \Delta x \left(g(x_i) - f(x_i)\right)}$$

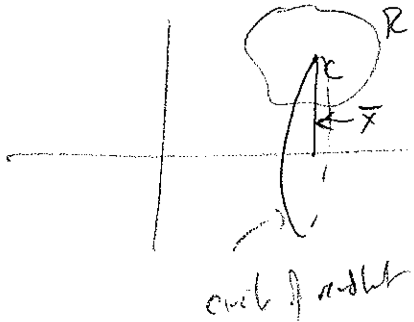
$$\bar{x} = \lim_{n \to \infty} \hat{\ } = \frac{\rho \int_a^b x(g(x) - f(x))\, dx}{\rho \int_a^b g(x) - f(x)\, dx} = \frac{\int_a^b x(g(x) - f(x))\, dx}{A}$$

A = area.

$$\bar{y} \approx \frac{\displaystyle\sum_{i=1}^{n} \overbrace{\rho \left(\Delta x \left(g(x_i) - f(x_i)\right)\right)\left(\frac{g(x_i) + f(x_i)}{2}\right)}^{\text{moment wrt } x}}{\displaystyle\sum_{i=1}^{n} \rho \Delta x \left(g(x_i) - f(x_i)\right)}$$

$$\bar{y} = \frac{\int_a^b \frac{1}{2}\left(g(x)^2 - f(x)^2\right) dx}{A}$$

<u>Pappus Theorem</u>: R = plane region, C = centroid, R on one side of x-axis. Then volume of revolution = Area(R) $\cdot$ length of circle of revolution of C.



R as above: $g(x) \geq f(x)$ on $[a,b]$, then

$$\text{Volume} = 2\pi \int_a^b x\left(f(x) - g(x)\right) dx$$

$$= 2\pi(\bar{x}A) = 2\pi \bar{x} \cdot A$$