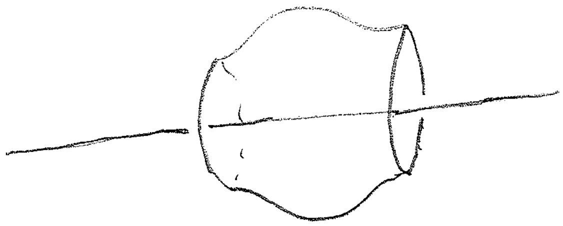
Math 231
2/11/11

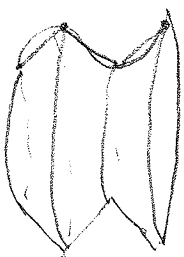
Surface area:

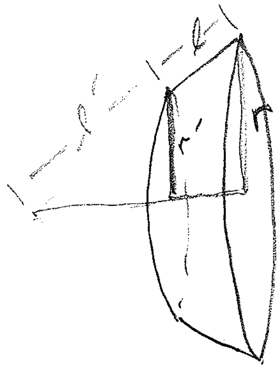
surface of revolution: curve disjoint from a line
then revolve around the line
in space



surface area = ? — approximate & take a limit:

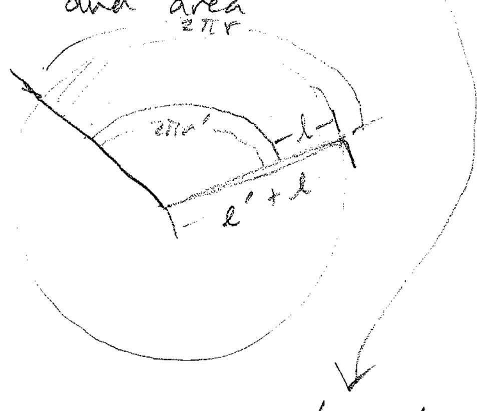
What do we approximate with? — approximate curve by segments, rotate these





Observe $\frac{r}{l'+l} = \frac{r'}{l'}$

and area



$$\text{Area} = \frac{r\pi(l'+l)^2}{l'+l} - \frac{r'\pi l'^2}{l'}$$

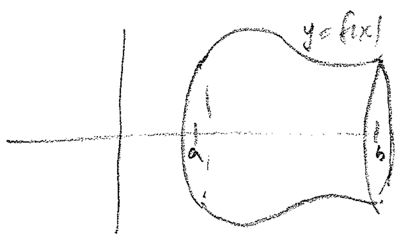
$$= \pi\left(r(l'+l) - r'l'\right) = \pi\left(rl + l'(r-r')\right)$$

$$= \pi(rl + r'l) = 2\pi l\left(\underbrace{\frac{r+r'}{2}}_{\text{average of } r,\, r'}\right)$$

$rl' - r'l' - r'l = 0$

$(r-r')l' = r'l$

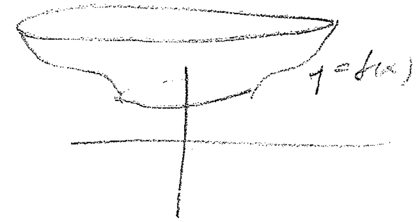
suppose curve = graph of $f(x)$, revolving around x-axis



$n \Rightarrow \Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$

$$\text{Area} \approx \sum_{i=1}^{n} 2\pi\left(\frac{f(x_{i-1}) + f(x_i)}{2}\right)\sqrt{\Delta x^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$\approx \sum_{i=1}^{n} 2\pi\, f(x_i^*)\sqrt{1 + (f'(x_i^*))^2}\,\Delta x$$



So $\text{Area} = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$

What if we revolve $y=f(x)$ around y-axis, assuming $0<a<b$?

$$\text{Area} \approx \sum_{i=1}^{n} 2\pi\left(\frac{x_{i-1}+x_i}{2}\right)\sqrt{1+(f'(x_i^*))^2}\,\Delta x \approx \sum_{i=1}^{n} 2\pi x_i^*\sqrt{1+(f'(x_i^*))^2}\,dx$$

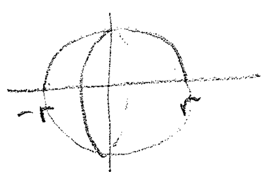
$$\text{So Area} = \int_a^b 2\pi x\sqrt{1+(f'(x))^2}\,dx.$$

from this easy to derive revolution of $x=g(y)$ around either axis:

x-axis $\displaystyle\int_c^d 2\pi y\sqrt{1+(g'(y))^2}\,dy$

y-axis $\displaystyle\int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy$

EX Check on a sphere:



$x^2+y^2=r^2$

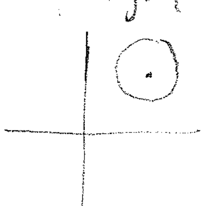
$y=\sqrt{r^2-x^2}$ $\quad -r\le x\le r$, revolve around x-axis:

$$f'(x) = \frac{-2x}{2\sqrt{r^2-x^2}} = \frac{-x}{\sqrt{r^2-x^2}}$$

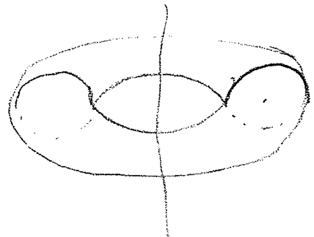
$$\int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\,\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = 2\pi\int_{-r}^{r}\sqrt{r^2-x^2+x^2}\,dx = 2\pi r\int_{-r}^{r}dx = 2\pi r\,x\Big|_{-r}^{r}$$
$$= 4\pi r^2$$

As with arc length, the integrals are often difficult to compute

EX Write integral to compute surface area of surface obtained by revolving the circle
[Student]



$(x-2)^2+y^2=1$ about the y-axis.

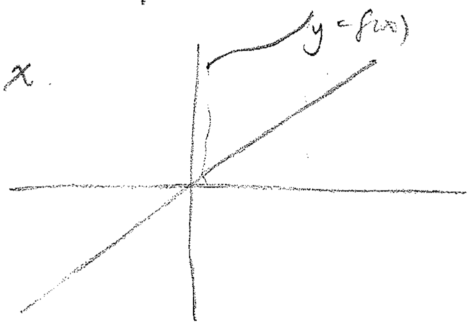


$y = f(x) = \sqrt{1-(x-2)^2} = \sqrt{2x - x^2 - 3}$

$f'(x) = \dfrac{-(x-2)}{\sqrt{1-(x-2)^2}}$

$$\int_1^3 2\pi x \sqrt{1 + \frac{(x-2)^2}{1-(x-2)^2}}\, dx = \int_1^3 2\pi x \sqrt{\frac{1+(x-2)^2}{1-(x-2)^2}}\, dx$$

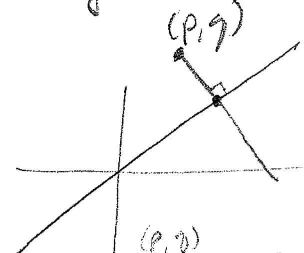
Students [in groups?] Suppose graph $y=f(x)$ for $a \le x \le b$ does not intersect the line $y = x$.



Find an integral which compute surface area of surface of revolution obtained by revolving graph $y=f(x)$ around line $y=x$.

Assume, $f(x) > x$

1^{st} How far is (p,q) to the line $y=x$



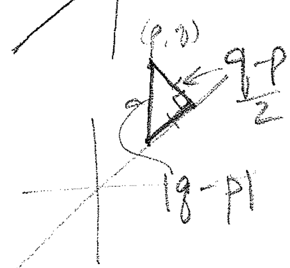
$$\begin{aligned} y &= (p+q) - x \\ + \quad y &= x \\ \hline 2y &= p+q \\ y &= \frac{p+q}{2} = x \end{aligned}$$

$$\text{dist} = \sqrt{\left(p - \frac{p+q}{2}\right)^2 + \left(q - \frac{p+q}{2}\right)^2}$$

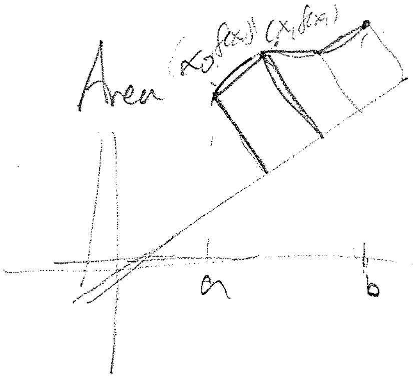
$$= \sqrt{\left(\frac{p-q}{2}\right)^2 + \left(\frac{q-p}{2}\right)^2}$$

$$= \sqrt{2\,\frac{(p-q)^2}{4}} = \boxed{\frac{|p-q|}{\sqrt{2}}}$$

or



Area



$$\text{Area} \approx \sum_{i=1}^{n} 2\pi \frac{|f(x_i^*) - x_i^*|}{\sqrt{2}} \sqrt{1 + (f'(x_i^*))^2}\, \Delta x$$

$$= \int_a^b \sqrt{2}\,\pi (f(x) - x) \sqrt{1 + (f'(x))^2}\, dx$$