Math 231
1-28-11

Integrating rational functions.

rational function : $\dfrac{\text{polynomial}}{\text{polynomial}}$

Eg $\dfrac{x^2+2x-1}{x^3+x^2-5}$, $\dfrac{x-1}{x^2-13x+2}$, $\dfrac{x^6-4}{x^3-2x+1}$, $\dfrac{P(x)}{Q(x)}$

Goal: describe "algorithm" to compute $\int \frac{P(x)}{Q(x)}dx$, for any rational $\frac{P(x)}{Q(x)}$.
[often practical, theoretically always possible — requires ability to factor polynomials into linear & quadratic factors]

Special cases:

$a,b \in \mathbb{R}$, $a \neq 0$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{du}{u} = \frac{1}{a}\ln|u| = \frac{1}{a}\ln|ax+b| + C$$

$u = ax+b$
$du = a\,dx$

$n>1$, integer

$$\int \frac{dx}{(ax+b)^n} = \frac{1}{a}\int \frac{du}{u^n} = \frac{1}{a}\frac{1}{(1-n)u^{n-1}} + C = \frac{1}{a(1-n)(ax+b)^{n-1}} + C$$

$u = ax+b$
$du = a\,dx$

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ax^2+bx+c — irreducible quadratic : can't be factored over $\mathbb{R}$, equivalently $b^2-4ac<0$. [quadratic eqn]

complete the square:

$$ax^2+bx+c = \left(ax^2+bx+\frac{b^2}{4a}\right)+c-\frac{b^2}{4a} = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a} \quad (4ac-b^2>0)$$

$$= a\left((x+d)^2+t^2\right) \qquad t^2 = \frac{4ac-b^2}{4a^2}$$

$$t = \frac{\sqrt{4ac-b^2}}{2a}$$

then

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{(x+d)^2+t^2} = \frac{1}{a}\int \frac{du}{u^2+t^2} = \frac{1}{at}\tan^{-1}\left(\frac{u}{t}\right)+C = \text{substitute back in}$$

$u = x+d$
$du = dx$

$$\left[= \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{\left(x+\frac{b}{2a}\right)2a}{\sqrt{4ac-b^2}}\right)+C \right]$$

more generally, $k > 1$

$$\int \frac{dx}{(ax^2+bx+c)^k} = \frac{1}{a^k t^k}\int \frac{du}{(u^2+1)^k} = \frac{1}{a^k t^k}\int \frac{\sec^2\theta \, d\theta}{\sec^{2k}(\theta)}$$

$$= \frac{1}{a^k t^k}\int \cos^{2k-2}(\theta)\, d\theta \quad \text{can solve.}$$

$$\int \frac{x\,dx}{(x^2+t^2)^k} = \frac{1}{2}\int \frac{du}{u^k} = \frac{1}{2(1-k)u^{k-1}} + C = \frac{1}{2(1-k)(x^2+t^2)^{k-1}}$$

$u = x^2+t^2$
$du = 2x\,dx$

We can reduce any rational function to sums of the kinds just discussed and polynomials (theoretically) — <u>partial fractions</u>.

<u>Step I</u>: long division, make degree numerator < degree denominator

$$\frac{x^3+x^2+1}{x^2-1} = x+1 + \frac{x+2}{x^2-1}$$

$$\begin{array}{r} x+1 \\ x^2-1 \,\big)\, \overline{x^3+x^2+0x+1} \\ -(x^3 \quad\; - x) \\ \hline x^2+x+1 \\ -(x^2 \quad\; -1) \\ \hline x+2 \end{array}$$

$$\left[\begin{array}{c} x^3+x^3+1 - (x+1)(x^2-1) = x+2 \\ \text{now divide by } x^2-1 \end{array}\right]$$

<u>Step 2</u> factor denominator in remaining fraction.

$$\frac{x^3+x^2+1}{x^2-1} = x+1 + \frac{x+2}{(x-1)(x+1)}$$

Step 3 express remaining fraction as sum of partial fractions

these have the form $\frac{A}{(ax+b)^k}$ or $\frac{Ax+B}{(ax^2+bx+c)}$ with ax^2+bx+c irreducible quadratic.

$$\frac{x+2}{(x-1)(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

only linear factors, not repeated

multiply these out, solve for A_1, A_2:

$$\frac{x+2}{(x-1)(x+1)} = \frac{A_1(x+1)+A_2(x-1)}{(x-1)(x+1)} \Rightarrow x+2 = (A_1+A_2)x + A_1 - A_2$$

$$\Rightarrow A_1 + A_2 = 1 \quad \text{(coeff of } x\text{)}$$

$$A_1 - A_2 = 2 \quad \text{(constant term)}$$

$$2A_1 = 3 \Rightarrow A_1 = 3/2$$

$$\tfrac{3}{2} - A_2 = 2 \Rightarrow A_2 = \tfrac{3}{2} - 2 = \tfrac{-1}{2}$$

So

$$\frac{x+2}{(x-1)(x+1)} = \frac{3/2}{x-1} - \frac{1/2}{x+1}$$ and

$$\frac{x^3+x^2+1}{x^2-1} = x+1+\frac{3/2}{x-1} - \frac{1/2}{x+1}.$$

Other cases: repeated linear factors:

$$\frac{4x^2+13x}{(x+2)^2(x-3)} = \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2} + \frac{A_3}{x-3}$$

[allow all powers of linear factors, up to highest power appearing

$$\Rightarrow \frac{4x^2+13x}{(x+2)^2(x-3)} = \frac{A_1(x+2)(x-3)+A_2(x-3)+A_3(x+2)^2}{(x+2)^2(x-3)}$$

$$4x^2+13x = A_1x^2 - A_1x - 6A_1 + A_2x - 3A_2 + A_3x^2 + 4A_3x + 4A_3$$

x^2 coeff: $4 = A_1 + A_3$ $\Rightarrow A_1 = 4 - A_3$

x coeff: $13 = -A_1 + A_2 + 4A_3$ $\quad 13 = A_3 - 4 + A_2 + 4A_3$

const: $0 = -6A_1 - 3A_2 + 4A_3$ $\quad 0 = -6(4-A_3) - 3A_2 + 4A_3$

$\Rightarrow 17 = A_2 + 5A_3$, $\quad 24 = -3A_2 + 10A_3$

$\Rightarrow 24 - 34 = (-3-2)A_2 \Rightarrow -10 = -5A_2 \Rightarrow A_2 = 2 \Rightarrow 17 - 2 = 5A_3 \Rightarrow A_3 = 3$

$\Rightarrow A_1 = 4 - 3 = 1$

So

$$\frac{4x^2+13x}{(x+2)^2(x-3)} = \frac{1}{x+2} + \frac{2}{(x+2)^2} + \frac{3}{(x-3)}.$$

For irred. quadratics, it gets messier, especially with repeated factors

$$\frac{9x^3+x^2-1}{(x^2+2)^2(x^2+x+1)} = \frac{A_1x+B_1}{x^2+2} + \frac{A_2x+B_2}{(x^2+2)^2} + \frac{A_3x+B_3}{x^2+x+1}$$

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compute $\int \sec x\,dx = \int \frac{1}{\cos x}dx = \int \frac{\cos x\,dx}{\cos^2 x} = \int \frac{\cos x\,dx}{1-\sin^2 x}$

$= \int \frac{du}{1-u^2} = \frac{1}{2}\int \frac{1}{1+u} + \frac{1}{1-u}\,du = \frac{1}{2}\ln|1+u| - \frac{1}{2}\ln|1-u|$

$u = \sin x$

$du = \cos x\,dx$

EX use this technique to integrate $\int \sec x\,dx = \int \frac{\cos x}{\cos^2 x}dx = \int \frac{du}{1-u^2}$

$u = \sin x$

$du = \cos x\,dx$

$= \frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u}\,du = \frac{1}{2}\left(\ln|1+u| - \ln|1-u|\right) + C$

$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + C = \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C$

$= \ln|\sec x + \tan x| + C$