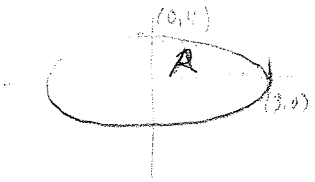
Math 231
1-26-11

trig substitution.

Suppose we want to compute area of region bounded by an ellipse:



$\frac{x^2}{9}+\frac{y^2}{4}=1$

$y^2=4\left(1-\frac{x^2}{9}\right)$

top arc: $y=2\sqrt{1-x^2/9}$

Area = 4 Area(R)

$=4\int_0^3 2\sqrt{1-x^2/9}\,dx = 8\int_0^3\sqrt{1-\left(\frac{x}{3}\right)^2}\,dx$

How to compute this?

trick if $\frac{x}{3}=\sin\theta$, then $1-\left(\frac{x}{3}\right)^2=\cos^2\theta$, and $\sqrt{1-\left(\frac{x}{3}\right)^2}=\sqrt{\cos^2\theta}=|\cos\theta|$

then $dx=3\cos\theta\,d\theta$, hence

$$\int\sqrt{1-\left(\tfrac{x}{3}\right)^2}\,dx=\int\sqrt{1-\sin^2\theta}\;3\cos\theta\,d\theta=3\int|\cos\theta|\cos\theta\,d\theta.$$

limits of integration?

x goes from 0 to 3 so $3\sin\theta$ goes from 0 to 3, hence θ goes from 0 to $\pi/2$ [for example]

thus

$$8\int_0^3\sqrt{1-(x/3)^2}\,dx=24\int_0^{\pi/2}|\cos\theta|\cos\theta\,d\theta=24\int_0^{\pi/2}\cos\theta\cos\theta\,d\theta=24\int_0^{\pi/2}\cos^2\theta\,d\theta=$$

$\cos\theta\geq 0$ on $[0,\pi/2]$

$$=24\left(\frac{\theta}{2}+\frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2}=24\cdot\frac{\pi}{4}=6\pi.$$

What's going on? two points of view:

(1) $x=3\sin\theta$, $x\in[0,3]$ $\longleftrightarrow$ $\sin^{-1}\left(\frac{x}{3}\right)=\theta$ $\theta\in[0,\pi/2]$

↳ Now do this substitution into $\int\sqrt{1-\left(\frac{x}{3}\right)^2}\,dx$ relating dx & $d\theta$ by $dx=3\cos\theta\,d\theta$

(2) "reverse substitution procedure"

$$\int_0^{\pi/2}\cos^2\theta\,d\theta=\int_0^{\pi/2}\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta=\frac{1}{3}\int_0^3\sqrt{1-(x/3)^2}\,dx$$

$x=3\sin\theta$
$dx=3\cos\theta\,d\theta$

More generally; integrals involving $\sqrt{a^2-x^2}$, $a>0$ can be computed using this substitution $x = a\sin\theta$

Observe: $-a \le x \le a$ for $\sqrt{a^2-x^2}$ to be defined, thus we want $-\pi/2 \le \theta \le \pi/2$. [sin maps $[-\pi/2, \pi/2]$ one-to-one onto $[-a, a]$.]

then $\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = a\sqrt{1-\sin^2\theta} = a|\cos\theta|$

because $-\pi/2 \le \theta \le \pi/2$, we have $|\cos\theta| = \cos\theta$

<u>EX</u>

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx = \int \frac{8\sin^3\theta \cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}} = 4\int \frac{\sin^3\theta\cos\theta}{\cos\theta}\,d\theta$$

$x = 2\sin\theta$
$dx = 2\cos\theta\, d\theta$

$[\cos(\theta) = |\cos\theta| \text{ in } (-\pi/2, \pi/2)]$

$$= 4\int \sin^2\theta \sin\theta\, d\theta = -4\int 1-u^2\, du = 4\left(\frac{u^3}{3} - u\right) + C$$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$

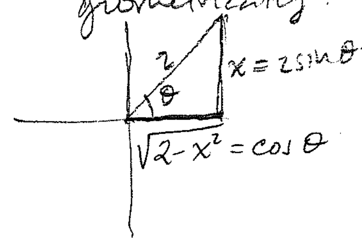
$$= 4\left(\frac{\cos^3\theta}{3} - \cos\theta\right) + C = 4$$

$\boxed{\text{In terms of } x\text{?}}$

$$= 4\left(\frac{\cos^3(\sin^{-1}(\frac{x}{2}))}{3} - \cos(\sin^{-1}(\tfrac{x}{2}))\right) + C \qquad \theta = \sin^{-1}\left(\tfrac{x}{2}\right)$$

$$= 4\left(\frac{(\sqrt{2-x^2})^3}{3} - \sqrt{2-x^2}\right)$$

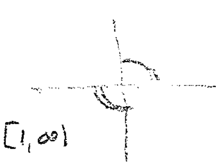
geometrically:



<u>similarly</u>:

$\sqrt{a^2+x^2} \rightsquigarrow x = a\tan\theta \qquad \theta \in (-\pi/2, \pi/2) \qquad$ <u>identity</u> $\tan^2\theta + 1 = \sec^2\theta$

$\sqrt{x^2-a^2} \rightsquigarrow x = a\sec\theta \qquad \theta \in [0, \pi/2)$ or $[\pi, 3\pi/2)$



same identity $\qquad$ sec maps $[0, \pi/2)$ to $[1, \infty)$
$[\pi, 3\pi/2)$ to $(-\infty, -1]$

Ex
[student]

$$\int \frac{x^3}{\sqrt{x^2+100}}\,dx = 1000\int \frac{\tan^3\theta \; 10\sec^2\theta\, d\theta}{\sqrt{100(\tan^2\theta+1)}} = 1000\int \frac{\tan^3\theta \sec^2\theta\, d\theta}{\sqrt{\sec^2\theta}}$$

$x = 10\tan\theta$
$dx = 10\sec^2\theta\, d\theta$

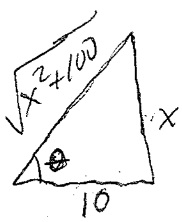
$\sec\theta > 0$ on $(-\pi/2, \pi/2)$

$$= 1000\int \tan^3\theta \sec\theta\, d\theta$$

$$= 1000\int \tan^2\theta \; \tan\theta\sec\theta\, d\theta$$

$$= 1000\int (\sec^2\theta - 1)\tan\theta\sec\theta\, d\theta = 1000\int u^2 - 1\, du$$

$u = \sec\theta$
$du = \sec\theta\tan\theta\, d\theta$



$$= 1000\left(\frac{u^3}{3} - u\right) + C = 1000\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C$$

$$= 1000\left(\frac{(\sqrt{x^2+100})^3}{3000} - \frac{1}{10}\sqrt{x^2+100}\right) + C$$

$$= \frac{(\sqrt{x^2+100})^3}{3} - 100\sqrt{x^2+100} + C$$

$\sec\theta = \frac{1}{10}\sqrt{x^2+100}$

One more trick here:

$$\int \sqrt{24-2x-x^2}\,dx = \int \sqrt{25-(x+1)^2}\,dx \underset{\substack{u=x+1\\ du=dx}}{=} \int \sqrt{25-u^2}\,du$$

$$24-2x-x^2 = 24-(x^2+2x+1)+1$$
$$= 25-(x+1)^2$$

$u = 5\sin\theta$
$du = 5\cos\theta\, d\theta$,

$$= \int \sqrt{25-25\sin^2\theta}\; 5\cos\theta\, d\theta = 25\int \sqrt{1-\sin^2\theta}\cos\theta\, d\theta = 25\int \cos^2\theta\, d\theta$$

$$= 25\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right) + C$$

$$= 25\left(\frac{\sin^{-1}(u)}{2} + \frac{1}{4}\sin(2\sin^{-1}(u))\right) + C$$

$$= 25\left(\frac{\sin^{-1}(x+1)}{2} + \frac{1}{4}\sin(2\sin^{-1}(x+1)\right) + C$$