1/24/11

Integration of trig functions:

basic idea = trig identities:

<u>Ex</u>

$$\int \sin^5 x\,dx = \int \sin^4 x \sin x\,d = \int (1-\cos^2 x)^2 \sin(x)\,dx$$

$$= \int (1-2\cos^2 x + \cos^4 x)\sin x\,dx = -\int (1-2u^2+u^4)\,du$$

$u = \cos x$
$du = \sin x\,dx$

$$= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C$$

<u>Ex</u> [student]

$$\int \cos^3 x \sin^2 x\,dx = \int (1-\sin^2 x)\sin^2 x \cos x\,dx$$

$u = \sin x$
$du = \cos x\,dx$

$$= \int (1-u^2)u^2\,du = \int u^2 - u^4\,du = \frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + C$$

Similar w/ $\sec x$ & $\tan x$ using
$\sec^2 x = 1 + \tan^2 x$

<u>Ex</u>

$$\int \tan^2 x \sec^4 x\,dx = \int \tan^2 x(1+\tan^2 x)\sec^2 x\,dx = \int u^2(1+u^2)\,du$$

$u = \tan x$
$du = \sec^2 x\,dx$

$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

<u>Ex</u>

$$\int \tan^3 x \sec^3 x\,dx = \int \tan^2 x \sec^2 x \tan x \sec x\,dx = \int (\sec^2 x - 1)\sec^2 x \tan x \sec x\,dx$$

$u = \sec x$
$du = \tan x \sec x\,dx$

$$= \int (u^2-1)u^2\,du = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

<u>So far</u>: $\int \sin^m x \cos^n x\,dx$ $m,n \geq 0$, not both even, $u = \cos x$, or $\sin x$ (also valid for other odd powers)

$\int \tan^m x \sec^n x\,dx$ $m,n \geq 0$, $\begin{cases} n \text{ even} \Rightarrow \sec^{2k} x = (1+\tan^2 x)^{k-1}\sec^2 x \\ m \text{ odd} \Rightarrow \tan^{2k+1} x \sec^n x = (\sec^2 x + 1)^k \sec^{n-1} x \sec x \tan x \end{cases}$

can similarly attack $\int \csc^m x \cot^n x\,dx$ Ex[standard] $\int \csc^4 x \cot^4 x\,dx = \int \cot^4 x(1+\cot^2 x)\csc^2 x\,dx$

$\hookrightarrow = \int u^4(1+u^2)\,du = \frac{\cot^5 x}{5} + \frac{\cot^7 x}{7} + C$

$u = \cot x$

$du = -\csc^2 x\,dx$

These use the identities $\sin^2 x + \cos^2 x = 1$ and $\sec^2 x = 1 + \tan^2 x$

What about

$$\int \sin^2 x\,dx = ?$$

Recall $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, so

$$\int \sin^2 x\,dx = \frac{1}{2}\int 1-\cos 2x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

TRIG IDENTITIES:

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\sin(x+y) = \cos x \sin y + \cos y \sin x$.

How to remember these.

$e^{ix}e^{iy} = e^{i(x+y)} \Rightarrow (\cos x + i\sin x)(\cos y + i\sin y)$

$\cos x\cos y - \sin x\sin y + i(\cos x \sin y + \cos y \sin x)$

$= e^{ix}e^{iy} = e^{i(x+y)}$

$= \cos(x+y) + i\sin(x+y)$

$\mathbb{C}$-numbers are equal iff they have same real and imaginary parts.

..done!! [— you proved $e^{ix}e^{iy} = e^{i(x+y)}$ using these trig identities
so, should think of this as giving geometric meaning to this identity.]

special case $x=y$.

- $\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1}{2}(\cos 2x + 1)$

  or $= (1-\sin^2 x) - \sin^2 x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1}{2}(1-\cos 2x)$

- and $\sin 2x = 2\sin x \cos x$

Using these we can always compute $\int \sin^n x \cos^m x\,dx$ for any $n, m \geq 0$

Ex [student]

$$\int \sin^2 x \cos^2 x \, dx = \int \tfrac{1}{2}(1-\cos 2x)\tfrac{1}{2}(1+\cos 2x)\, dx$$

$$= \tfrac{1}{4}\int 1-\cos^2 2x \, dx = \tfrac{1}{4}x - \tfrac{1}{4}\int \tfrac{1}{2}(1+\cos 4x)\, dx$$

$$= \tfrac{1}{4}x - \tfrac{1}{8}x - \frac{1}{32}\sin 4x + C = \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C$$

alternative:

$$\int \sin^2 x \cos^2 x \, dx = \tfrac{1}{4}\int \sin^2 2x \, dx = \tfrac{1}{8}\int 1-\cos 4x \, dx = \tfrac{1}{8}x - \frac{1}{32}\sin 4x + C$$

$\int \tan x \, dx = \ln|\sec x| + C$ (write $\tan x = \frac{\sin x}{\cos x}$, do $u = \cos x$)

$\int \sec x \, dx = \ln|\sec x + \tan x| + C$ (write $\sec x = \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$, $u = \sec x + \tan x$)

another approach to this later: $\int \frac{1}{\cos x}\, dx = \int \frac{\cos x}{1-\sin^2 x}\, dx$

$= \int \frac{du}{1-u^2} = \ldots$ partial fractions.

Another family:

Ex $\int \sin mx \cos nx \, dx = \frac{1}{2}\int \sin((m-n)x) + \sin((m+n)x)\, dx$

$= -\frac{1}{2}\left(\frac{1}{m-n}\cos((m-n)x) + \frac{1}{m+n}\cos((m+n)x)\right) + C$

$\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$

$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

$e^{iA}e^{-iB} + e^{iA}e^{iB}$

$= e^{i(A-B)} + e^{i(A+B)}$

$\left(e^{iA}e^{-iB} - e^{iA}e^{iB} = e^{i(A-B)} - e^{i(A+B)}\right)$

& $\cos A = \cos -A$

$\sin -A = -\sin A$.