Techniques of integration Ch 7

Fundamental Theorem of Calculus:

differentiation and indefinite integration are inverse operations.

differentiation rules give rise to integration techniques.

f' $\quad\quad$ $\int f\,dx$

power rule $\longleftrightarrow$ power rule

sum rule $\longleftrightarrow$ sum rule

product rule $\longleftrightarrow$ substitution.

product rule $\longleftrightarrow$ Integration by parts

$$(fg)' = f'g + fg' \implies \int (fg)'(x)\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

or

$$\underbrace{\int f(x)g'(x)\,dx}_{\text{given}} = f(x)g(x) - \underbrace{\int f'(x)\,g(x)\,dx}_{\text{hope you can solve}}$$

or

$$\int \underbrace{u}_{f}\,\underbrace{dv}_{g'dx} = \underbrace{uv}_{fg} - \int \underbrace{v}_{g}\,\underbrace{du}_{f'dx}$$

EX $\displaystyle\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$

$u = x \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = e^x$

EX [students] $\displaystyle\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

$u = x \quad\quad du = dx$

$dv = \cos x\,dx \quad\quad v = \sin x$

This also allows us to compute

$$\int \ln x \, dx = x \ln x - \int x \cdot \tfrac{1}{x} dx = x \ln x - x + C$$

$u = \ln x \quad du = \frac{1}{x} dx$
$dv = dx \quad v = x$

sometimes we can apply integration by parts several times then solve for the integral.

Ex $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx = e^x \sin x - \left[-e^x \cos x - \int -e^x \cos x \, dx \right]$

$u = e^x \quad du = e^x dx$
$dv = \cos x \, dx \quad v = \sin x$

$u = e^x \quad du = e^x dx$
$dv = \sin x \, dx \quad v = -\cos x$

So

$$\int e^x \cos x \, dx = e^x (\sin x + \cos x) - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x (\sin x + \cos x)$$

$$\int e^x \cos x \, dx = \tfrac{1}{2} e^x (\sin x + \cos x) + C$$

What not to do:
Start same

$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

then set
$u = \sin x \quad du = \cos x \, dx$
$dv = e^x dx \quad v = e^x$

so

$\int e^x \cos x \, dx = e^x \sin x - \left[e^x \sin x - \int e^x \cos x \, dx \right]$

$\Rightarrow \int e^x \cos x \, dx = \int e^x \cos x \, dx$

Ex [Student] $\int e^{-x} \sin 2x \, dx = -\frac{e^{-x}}{2} \cos 2x - \int \frac{e^{-x}}{2} \cos 2x \, dx = -\left(\frac{e^{-x}}{2} \cos 2x + \left[\frac{e^{-x}}{4} \sin 2x + \frac{1}{4} \int e^{-x} \sin 2x \, dx \right] \right)$

$u = e^{-x} \quad du = -e^{-x} dx$
$dv = \sin 2x \, dx \quad v = -\frac{1}{2} \cos 2x$

$u = \frac{e^{-x}}{2} \quad du = -\frac{e^{-x}}{2} dx$
$dv = \cos 2x \, dx \quad v = \frac{1}{2} \sin 2x$

$$\Rightarrow \frac{5}{4} \int e^{-x} \sin 2x \, dx = -e^{-x} \left(\frac{\cos 2x}{2} + \frac{\sin 2x}{4} \right)$$

$$\int e^{-x} \sin 2x \, dx = -\frac{4}{5} e^{-x} \left(\frac{\cos 2x}{2} + \frac{\sin 2x}{4} \right) + C$$

Definite integrals:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Ex

$$\int_0^1 x^2 e^x dx = x^2e^x\Big|_0^1 - \int_0^1 2xe^x dx = e - 2\int_0^1 xe^x dx = e - 2\left[xe^x\Big|_0^1 - \int_0^1 e^x dx\right]$$

$u = x^2$ $du = 2x\,dx$
$dv = e^x dx$ $v = e^x$

$u = x$ $du = dx$
$dv = e^x dx$ $v = e^x$

$$= e - 2\left[e - e^x\Big|_0^1\right] = e - 2\left[e - (e-1)\right] = e - 2$$

useful trick: $p(x)$ a polynomial, $f(x)$ a function you can take antiderivatives of ~~forever~~ [you can compute antiderivative iteratively on $f(x)$]

$$\int p(x) f(x)\,dx = p(x)F(x) - \int p'(x)F(x)\,dx$$

$u = p(x)$ $du = p'(x)dx$
$dv = f(x)dx$ $v = F(x)$

Notice degree $p'(x)$ < degree $p(x)$.
highest power
iterating this eventually makes polynomial part go away.

Multiple integration techniques can be useful

Ex $\int \sin(\ln(x))\,dx = \int \sin(u)e^u du = \frac{1}{2}e^u(\sin u - \cos u) + C = \frac{1}{2}x(\sin \ln x - \cos \ln x) + C$

substitution
$u = \ln x$ $e^u = x$
$e^u du = dx$

compute using integration by parts

Reduction formulas [see book for more]

EX:
$$\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$$

$u = (\ln x)^n \qquad du = n(\ln x)^{n-1}\frac{1}{x}$

$dv = dx \qquad v = x$

Lots of different tricks/techniques
practice, practice, practice ...