Math 428: Tues, March 29 from 9:30–10:50 in 341 AH

Topic: Seifert surfaces for knots in $\mathbb{R}^3$.

Knot in $\mathbb{R}^3$: A closed polygonal loop K without self-intersections:

Seifert Surface: A polygonal orientable surface S in $\mathbb{R}^3$ where $\partial S = K$.

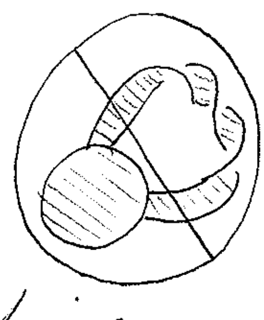
Ex:  Non ex: 

Def: A polygonal surface S is orientable if any of the following equivalent conditions hold:

ⓐ We can orient the triangles of S ( vs. ) so that along each edge the arrows run in opposite directions

(b) S does <u>not</u> contain a Möbius band.

(c) S has a handle decomposition where no 1-handle is twisted an odd # of times



(d) If $S \subseteq \mathbb{R}^3$ then there is a consistent choice of normal vectors. (i.e. is two-sided)

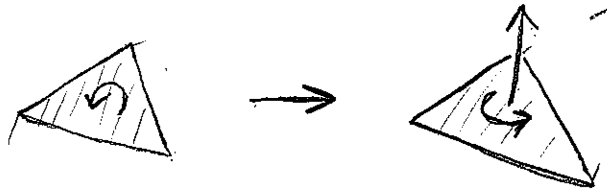
Exercise: Prove these are equivalent.

Hint:

$$\begin{tikzcd} (a) \arrow[r, Rightarrow] & (b) \arrow[r, Rightarrow] & (c) \arrow[ll, bend left, Rightarrow] \end{tikzcd}$$

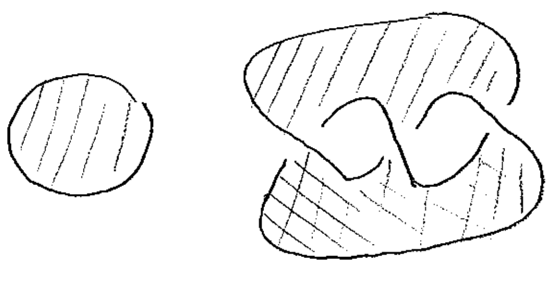
Assume any two triangulations have a common subdivision

(a) $\Longleftrightarrow$ (d) via the right-hand rule.



Thm: Every knot in $\mathbb{R}^3$ bounds a Seifert surface.

Ex:



What about  ?

Note  is non-orientable.

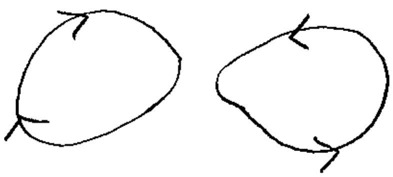
Seifert's Algorithm:

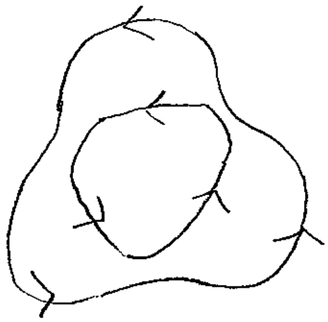
1. Start with a projection of K in $\mathbb{R}^2$, and orient it

(a)

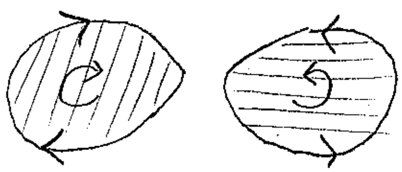
(b)

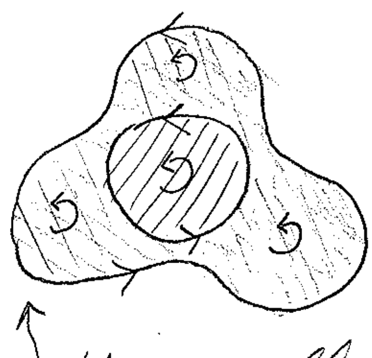
2. Erase the crossings, and connect up the resulting strands so the orients match. Get a bunch of "Seifert circles."





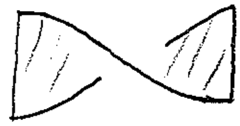
3. Fill in each circle with a disc. These can be nested

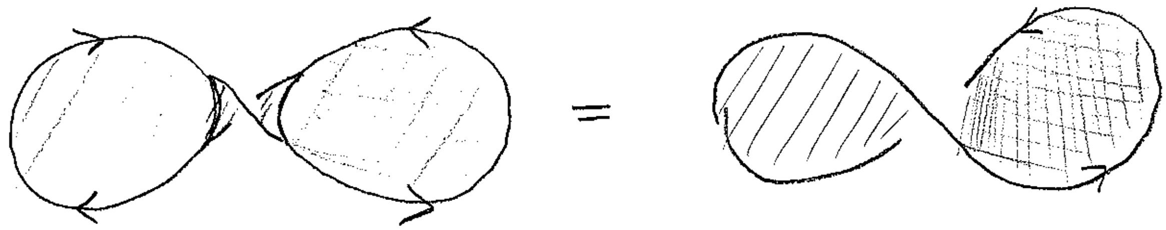


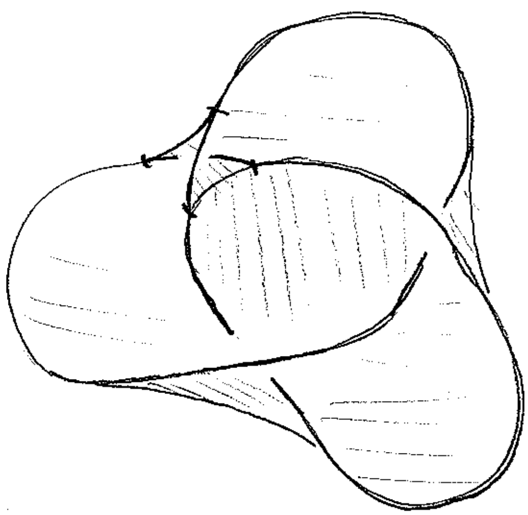


the smaller disc.

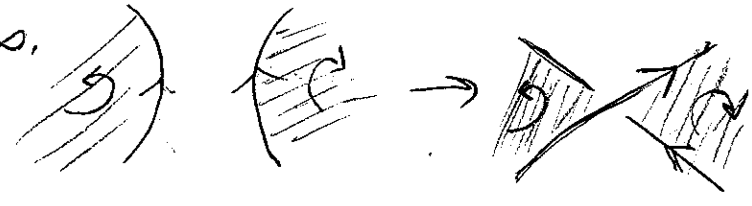
Orient each disc to match the boundary curves.

4. At each former crossing, glue in a twisted band  to recreate the knot.





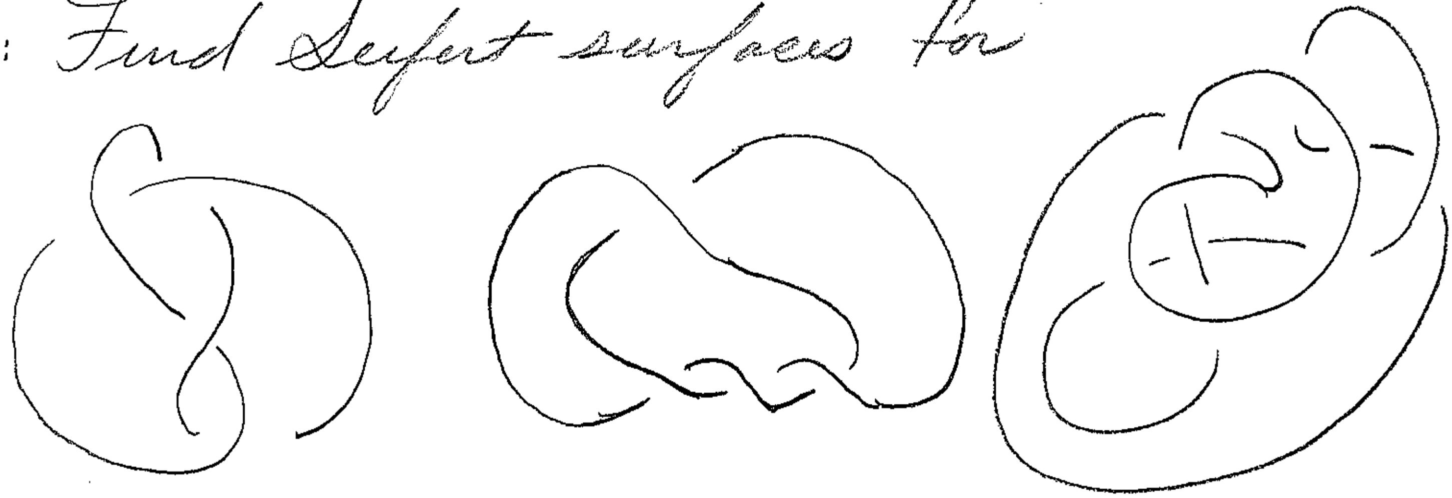
This gives a surface S with boundary K. It is always orientable, since the twists go between boundaries of discs with locally opposite orientations.



<u>Exercise:</u> Fill in the details for why S is orientable.

Note: This procedure proves the theorem.

Ex: Find Seifert surfaces for



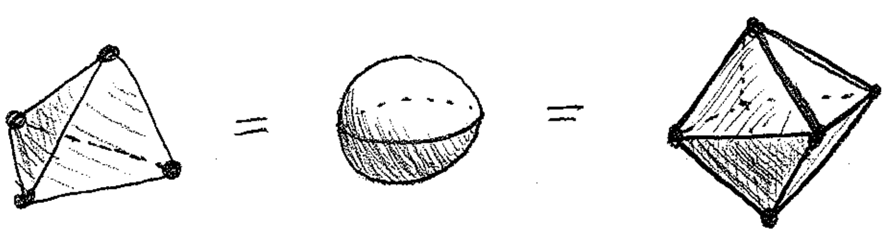
Basic Q: How do we tell surfaces apart?

Euler char: S a surface (with or without boundary) with a triangulation $\mathcal{T}$. Set

$$\chi(S,\mathcal{T}) = V - E + F$$

V = # verts in $\mathcal{T}$, E = # edges in $\mathcal{T}$, F = # triangles in $\mathcal{T}$.

Ex:



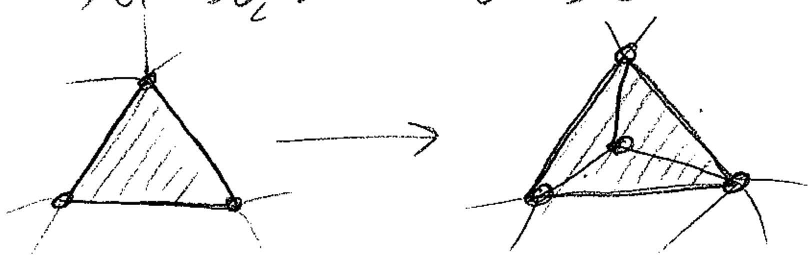
$\chi = 4-6+4$
$= 2$

$\chi = 6-12+8$
$= 2$

Thm: If $\mathcal{T}_1, \mathcal{T}_2$ are triangulations of S then $\chi(S,\mathcal{T}_1) = \chi(S,\mathcal{T}_2)$.

Proof:

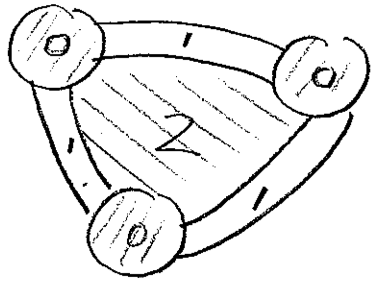
"Wrong": $\mathcal{T}_1$ and $\mathcal{T}_2$ have a common subdivision $\mathcal{T}'$. Check $\chi(S,\mathcal{T}_i) = \chi(S,\mathcal{T}')$. E.g.



$\chi = 3-3+1 = 1 = 4-6+3$

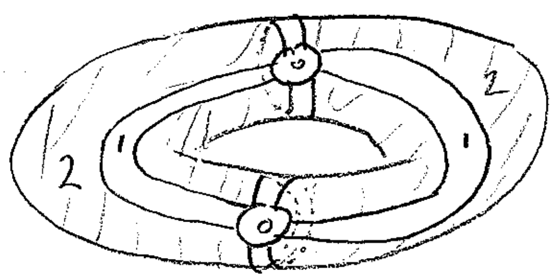
"Right:" Use (singular) homology or fundamental group, relate to combinatorics of $\mathcal{T}$.

Also works for handle decompositions where we allow any number of 0 and 2 handles.



$=$

$$\chi = 3-3+1 = 1$$



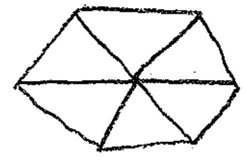
$=$

$$\chi = 2-4+2 = 0$$

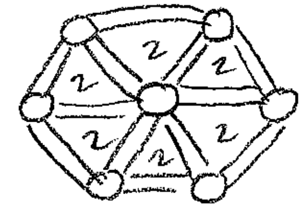
$$\chi(S,\mathcal{H}) = (\#\text{ of 0-handles}) - (\#\text{ of 1-handles}) + (\#\text{ of 2-handles})$$

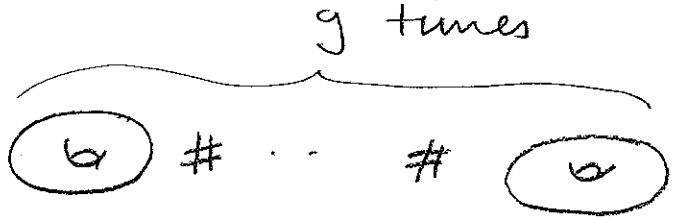
<u>Thm</u>: Does not depend on $\mathcal{H}$, same as if you used a triangulation.

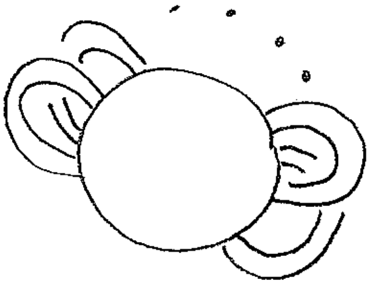
Just part is clear



$\longrightarrow$

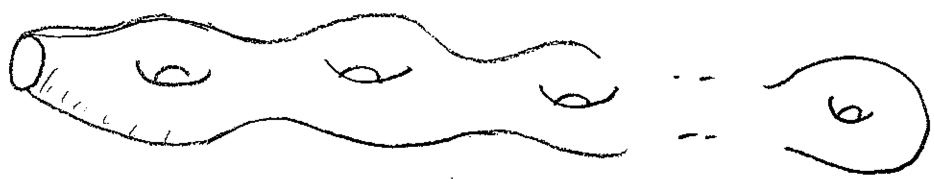


Ex: $S_g =$ 

$=$  $+ \, 2$ handle

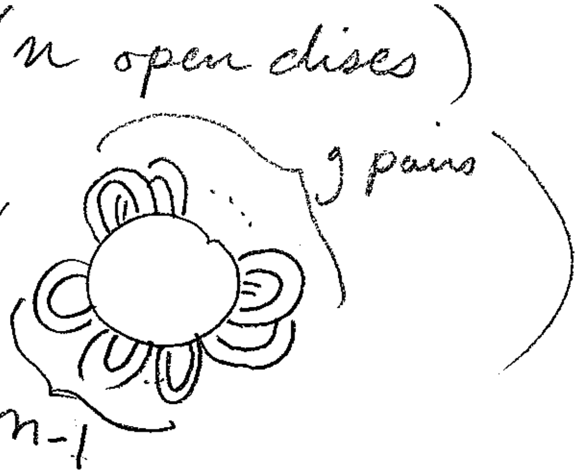
$$\chi(S_g) = 1 - 2g + 1 = 2 - 2g$$

Ex: $S_{g,1} = S_g \setminus$ open disc

$=$ 

$$\chi(S_{g,1}) = 1 - 2g$$

Ex: $S_{g,n} = S_g \setminus (n \text{ open discs})$

$\chi(S_{g,n}) = \chi\Big($  $\Big)$

$$= 1 - 2g - (n-1) = 2 - 2g - n$$

<u>Thm</u>: Any compact orientable surface is homeomorphic to $S_{g,n}$ for a unique (g,n).

<u>Proof</u>: Existence and manipulation of handle decomp prove the first part.

If $S_{g,n} \cong S_{g',n'}$ then must have $n=n'$ as the number of connected components of ∂S is clearly a topological invariant. Then the formulae from the previous page and invariance of Euler char force $g=g'$. ▨