Given a (Laurant) polynomial $p(x)$, define

$M\,p(x)$ = max degree of nonzero term of $p(x)$

$m\,p(x) \equiv$ min " " " " " "

Lemma II.15 Let D be a diagram with n crossings.

Then (i) $M\langle D\rangle \leq n + 2|S_+| - 2$ with equality if D is $+$adequate

(ii) $m\langle D\rangle \geq -n - 2|S_-| + 2$ " " " D is $-$adequate

proof Observe that

$$M(\langle D|S_+\rangle(-A^2-A^{-2})^{|S_+|-1}) = M(A^n(-A^2-A^{-2})^{|S_+|-1}) = n+2|S_+|-2$$

Since

$$\langle D\rangle = \sum_S \langle D|S\rangle(-A^2-A^{-2})^{|S|-1}$$

we just need to show that $M(\langle D|S\rangle(-A^2-A^{-2})^{|S|-1}) \leq n+2|S_+|-2$

with equality if D is $+$adequate.

For this, observe that an arbitrary state S is obtained from S_+ by a sequence of states $S_+ = S_1, \ldots, S_k = S$ w/ S_{j+1} having one more A^{-1}-resolution than S_j. Thus, we first show that $\forall j$

$$\langle D|S_j\rangle(-A^2-A^{-2})^{|S_j|-1} \geq \langle D|S_{j+1}\rangle(-A^2-A^{-2})^{|S_{j+1}|-1}$$

This is clear, since $\langle D|S_j\rangle = \sum_{i=1}^n S_j(i)$

$$\langle D|S_{j+1}\rangle = \sum_{i=1}^n S_{j+1}(i) = \sum_{i=1}^n S_j(i) - 2$$

since $S_{j+1} = S_j$ w/ one exception i and $S_{j+1}(i) = -1$, $S_j(i) = 1$

and $|S_{j+1}| = |S_j| \pm 1$.

So $\langle D|S_{j+1}\rangle$ term goes down by 2

$(-A^2-A^{-2})^{|S_{j+1}|}$ terms goes up or down by 2.

This proves the inequality since

$$n+2|S_+|-2 \geq M(\langle D|S\rangle(-A^2-A^{-2})^{|S_+|-1}) \geq M(\langle D|S_2\rangle(-A^2-A^{-2})^{|S_2|-1})$$
$$\geq \cdots \geq M(\langle D|S\rangle(-A^2-A^{-2})^{|S|-1})$$

Also, if D is +adequate, then $S_+=S_1$ and by defn.

$$|S_+|=|S_1|>|S_2|,$$ so very 1st inequality is strict.

This proves (i).

The proof of (ii) is similar $\square$

Corollary II.16 For any diagram D of a link L
$M\langle D\rangle - m\langle D\rangle \leq 2n+2|S_+|+2|S_-|-4$
with equality if D is adequate.

Lemma II.17 Let D be a connected link diagram w/ n crossings. Then

$$|S_+|+|S_-| \leq n+2$$

with equality if D is alternating.

Before we prove this, we need a fact about connected graphs in the plane.

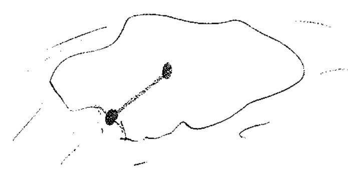
Proposition II.18 (baby Euler number) If $\Gamma \subset \mathbb{R}^2$ is a connected graph with V vertices, E edges and F complementary components, then $V-E+F=2$

<u>proof of proposition</u>: This can be proven by induction on the number of vertices, say.

A one-vertex graph is a point, and $V=1$, $E=0$, $F=1$ ✓

suppose true for $1 \le V \le n-1$, prove it for $V=n$.

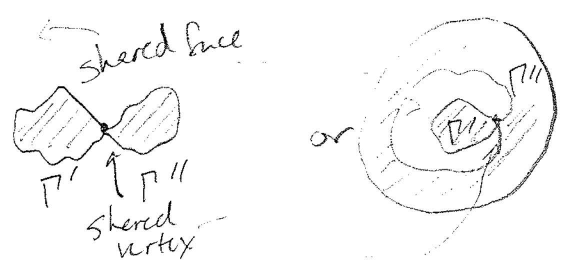
If graph Γ has a vertex with valence 1, it looks like



we can remove this vertex and the adjacent edge to produce a connected graph Γ' with

$$V(\Gamma') = V(\Gamma) - 1$$
$$E(\Gamma') = E(\Gamma) - 1 \qquad \text{so } V-E+F \text{ is the same for } \Gamma \text{ \& } \Gamma'!$$
$$F(\Gamma') = F(\Gamma)$$

If Γ has no valence 1 vertices, then look for a <u>cut vertex</u>: a vertex whose removal disconnects the graph into two nonempty pieces.



Let Γ', Γ'' be two subgraphs that meet in the cut vertex only and w/ $\Gamma = \Gamma' \cup \Gamma''$

By inspection, we have

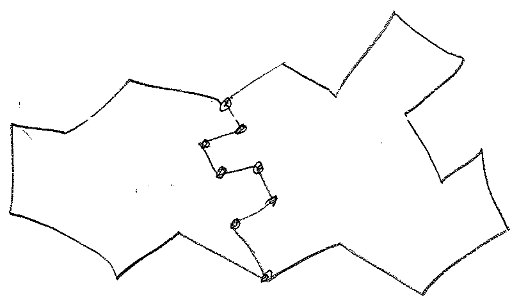
$$V(\Gamma) = V(\Gamma') + V(\Gamma'') - 1$$
$$E(\Gamma) = E(\Gamma) + E(\Gamma'')$$
$$F(\Gamma) = F(\Gamma') + F(\Gamma'') - 1$$

Then by induction

$$\begin{aligned} &V(\Gamma) - E(\Gamma) + F(\Gamma) \\ &= V(\Gamma') - E(\Gamma') + F(\Gamma') \\ &\quad + V(\Gamma'') - E(\Gamma'') + F(\Gamma'') \\ &\quad - 2 = 2 + 2 - 2 = 2. \end{aligned}$$

Finally, suppose no valence 1 vertex, no cut vertex.

Consider the circuit around the outside:



Γ

Either this is all of Γ, or there is an embedded edge path across the middle.

In first case, get $V = E$, $F = 2$ so $V - E + F = 2$ ✓

In 2^{nd} case, divide $\Gamma = \Gamma' \cup \Gamma''$, observe if edge path has k vertices, it has $k-1$ edges, so

$$V(\Gamma) = V(\Gamma') + V(\Gamma'') - k$$
$$E(\Gamma) = E(\Gamma') + E(\Gamma'') - k + 1$$
$$F(\Gamma) = F(\Gamma') + F(\Gamma'') - 1$$

and as before, induction kicks in and

$V(\Gamma) - E(\Gamma) + F(\Gamma) = 2$ ✓

$\square$

Now ready for

<u>Proof of Lemma II.17</u>: Induct on n. Obvious for $n = 0$, since a connected diagram w/ no crossings is $\bigcirc$, and $|S_+| = |S_-| = 1$

Suppose true for all connected diagrams w/ $\leq n-1$ crossings.

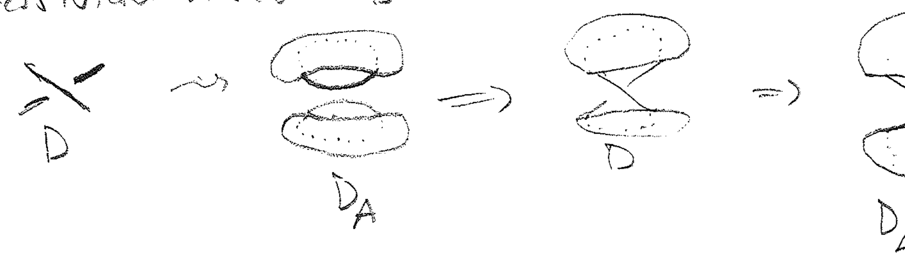
Let D be a connected diagram with n crossings.

write $S_+(D)$ and $S_-(D)$ for $+$ and $-$ states of D. (we'll have more than one diagram to consider)

Fix a crossing of D.

At least one of A or A^{-1} resolution keeps diagram connected

[If, e.g. A-resolution disconnects



]

Say A-resolution D_A is still connected. Then

$$|S_+(D_A)| + |S_-(D_A)| \le (n-1)+2 \quad \text{by inductive hypoth.}$$

Observe that

$$|S_+(D_A)| = |S_+(D)|$$

and

$$|S_-(D_A)| = |S_-(D)| \pm 1 \quad \text{— we changed one resolution.}$$

so

$$|S_+(D)| + |S_-(D)| = |S_+(D_A)| + |S_-(D_A)| \pm 1$$
$$\le ((n-1)+2)+1 = n+2, \text{ as required.}$$

Now we prove that get equality when D alternating.

By appropriate checker board coloring, $|S_+| = \#$ black regions, $|S_-| = \#$ white regions

Now consider D as just a connected graph. Observe that since all vertices of D are either 2-valent or 4-valent, if $n = \#$ of crossings $= \#$ 2-valent vertices, then $V-E=-n$

To see this, observe that every edge has 2 vertices, so $2E = 2(\#\text{2-valent vertices}) + 4(\#\text{4-valent vertices})$
$$= 2V + 2n$$

Therefore, since every face is either black or white, $F = |S_+| + |S_-|$ and
$$2 = V - E + F = -n + |S_+| + |S_-|$$
and so
$$|S_+| + |S_-| = n + 2. \quad \square$$

Define breadth of a Laurent polynomial $p(t)$, $B(p(t)) = Mp(t) - mp(t)$.

proof of Theorem II.13: It suffices to prove this for connected alternating diagrams — check this. Let D be any connected diagram for a link L. We have: $V_L(t) = (-A)^{-3w(D)} \langle D \rangle \big|_{t = A^{-4}}$

so $4B(V_L(t)) = B\langle D \rangle = M\langle D \rangle - m\langle D \rangle$
$$\leq 2n + 2|S_+| + 2|S_-| - 4 \leq 4n \quad \text{so } \boxed{BV_L(t) \leq n.}$$

If D is alternating, these are equalities by prev. lemmas, so $BV_L(t) = n$

If D is alternating and L had a diagram D' w/ $\leq n-1$ crossings then $\underset{n}{\overset{\shortparallel}{BV_L(t)}} \leq n-1$ $\downarrow\uparrow$. $\square$

Scholium II.19. For any oriented link w/ connected diagram having n crossings, we have $BV_L(t) \leq n$.