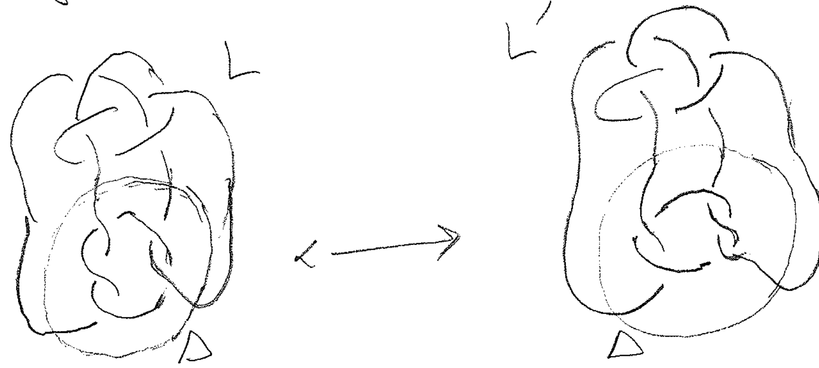
Open problem. Does there exist K a knot with $V_K(t) = 1$?

$V_K(t)$ is not a complete invariant: $\exists\, K \not\sim K'$ w/ $V_K(t) = V_{K'}(t)$ constructed from *mutation* as follows.

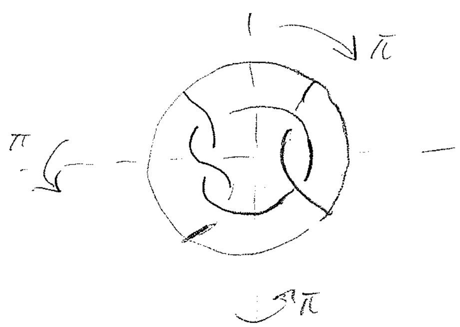
Let D be a diagram for L and suppose $\exists$ a disk Δ in projection plane with boundary circle meeting D in 4 points.



Kinoshita-Terasaka knot

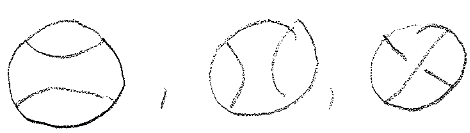
Conway knot.

a Mutation of L is a link L' obtained by one of 3 rotations of the diagram inside Δ keeping $D - \Delta$ fixed. [may need to change orientation on strands in Δ]



Proposition II.12: If L' is obtained from L by mutation, then $V_L(t) = V_{L'}(t)$.

Proof sketch: Using Prop. II.9, express $V_L(t)$ as $\mathbb{Z}[t^{1/2}, t^{-1/2}]$-linear combination of polynomials $V_{L_1}(t), \dots, V_{L_r}(t)$ s.t. each L_i in Δ looks like



possibly together with some disjoint circles.

This is achieved by iteratively changing crossings and resolving crossings within Δ. Now check that if we do the same to L', get $L_1', \dots, L_r'$ with $L_i = L_i'$, and applying Prop II.9 $V_{L'}(t)$ same linear combination of

$V_{L_1}(t), \ldots, V_{L_r}(t)$ — this is just because the crossing changes and resolutions commute with the rotation. Therefore

$V_L(t) = V_{L'}(t)$ ☐.

<u>Exercise II.14</u> Check this for the Kinoshita-Terasaka knot & Conway knot.

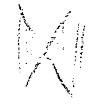
<u>Applications of Jones Polynomial</u> —

<u>Defn</u> Given a link L, the <u>crossing number</u> of L is the min. # of crossings in any diagram, write $C(L)$ = crossing # of L.

Clearly, $C(L)$ is a link invariant, and $C(L)=0$ iff L = unlink.

Great invariant, except generally impossible to compute.

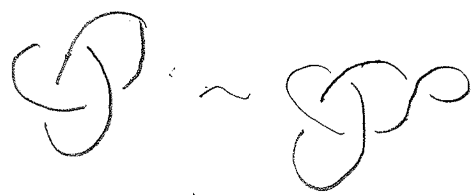
<u>Defn</u> The diagram D of a link L is called <u>alternating</u> if the crossings along any component of L alternate over/under, starting at any point.

Observe that for any projection one can choose crossings so that diagram is alternating. To see this, checkerboard color diagram, then choose crossings so that  occurs at each crossing (instead of ).

<u>Exercise II.15</u> ~~check that resulting diagram is alternating.~~

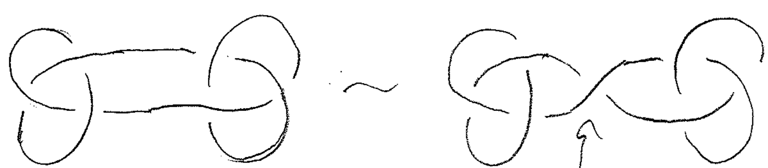
An alternating diagram is "the opposite" behavior to the unknot construction of always going under. One might expect/hope that alternating knots are "very knotted", & they are, with some obvious exceptional examples:

alternating.

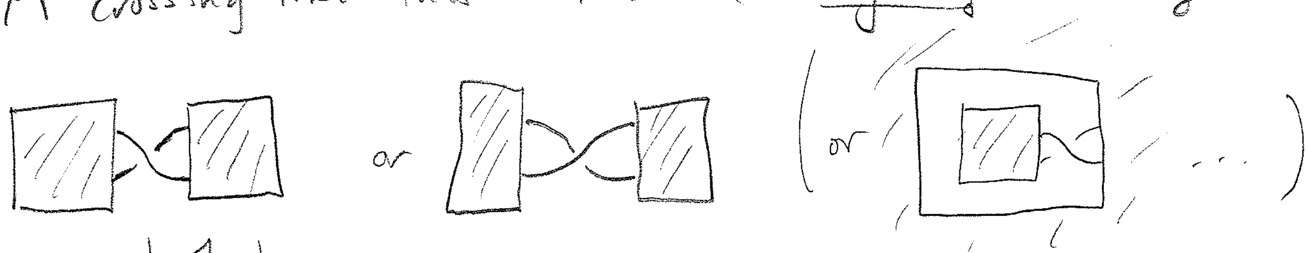


Can always add RI move (or reflection) and keep alternating property.

alternating.



A crossing like this is called a <u>nugatory</u> crossing



rest of diagram lies in boxes

<u>Defn</u> A diagram is called <u>reduced</u> if there are no nugatory crossings

<u>Theorem II. 13</u> If L is a link with a reduced alternating diagram with n crossings, then $c(L) = n$.

This theorem (proven by Kauffman, Murasugi, Thistlethwaite) answered an inherent question posed in late 1880's!
[try to prove it without Jones polynomial!]

Before proof, need some preliminary lemmas.

Recall a state S of a diagram D is a choice of A & B resolution of each crossing. Number the crossings $1, \dots, n$. Since $B = A^{-1}$, can view S as a map $S: \{1, \dots, n\} \longrightarrow \{\pm 1\}$, so that the resolution of i^{th} crossing determined by S is $A^{S(i)}$, recall $|S| = \#$ components after doing all resolution.

Proposition II.3 says

$$\langle D \rangle = \sum_S \left(A^{\sum_{i=1}^n S(i)} (-A^2 - A^{-2})^{|S|-1} \right) = \sum_S \langle D | S \rangle (-A^2 - A^{-2})^{|S|-1}$$

Let S_+, S_- be "constant" states — all A-resolutions and all A^{-1}-resolutions, respectively.

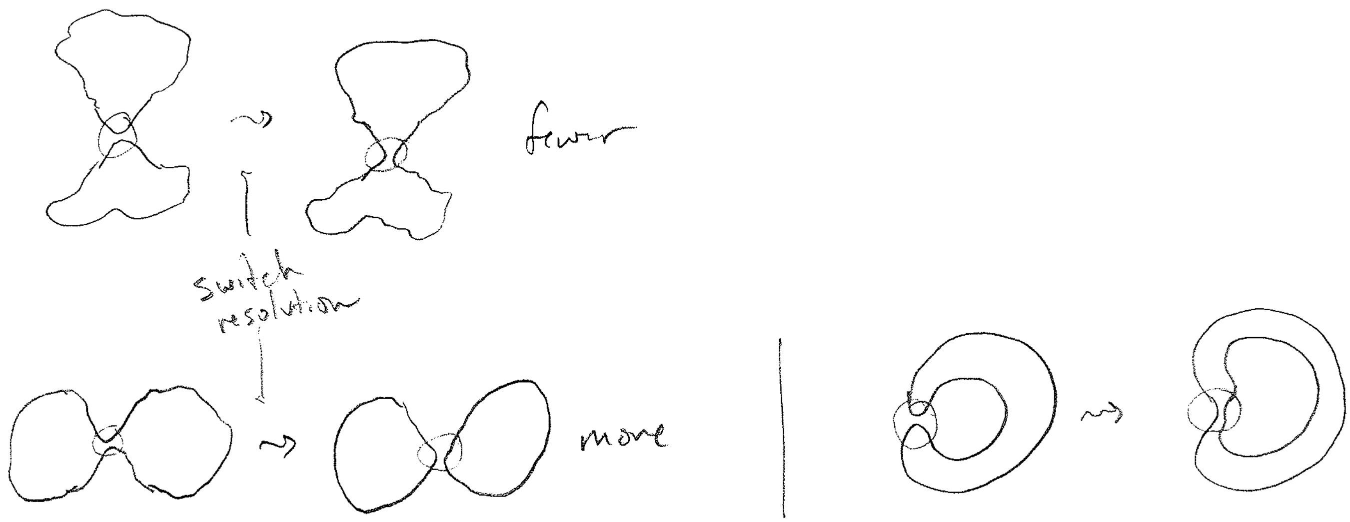
so $S_{\pm}(i) = \pm 1 \ \forall i$.

<u>Def'n</u> say D is <u>+adequate</u> if $|S_+| > |S| \ \forall$ states S with $\sum S(i) = n-2$ (so, a state with just one A^{-1} resolution)
Similarly D is <u>−adequate</u> if $|S_-| > |S| \ \forall$ states S with $\sum S(i) = 2-n$. D is adequate if both.

<u>Lemma II.14</u> A reduced alternating diagram is adequate.

<u>proof</u>: 1st observe that D is adequate iff each component of D is adequate, so assume D is connected.

Now, When is a diagram +adequate? To understand this first do all resolutions for S_+. Then you want to check whether switching any single resolution will result in more or fewer components — note: it either goes up or down, it can't stay the same.

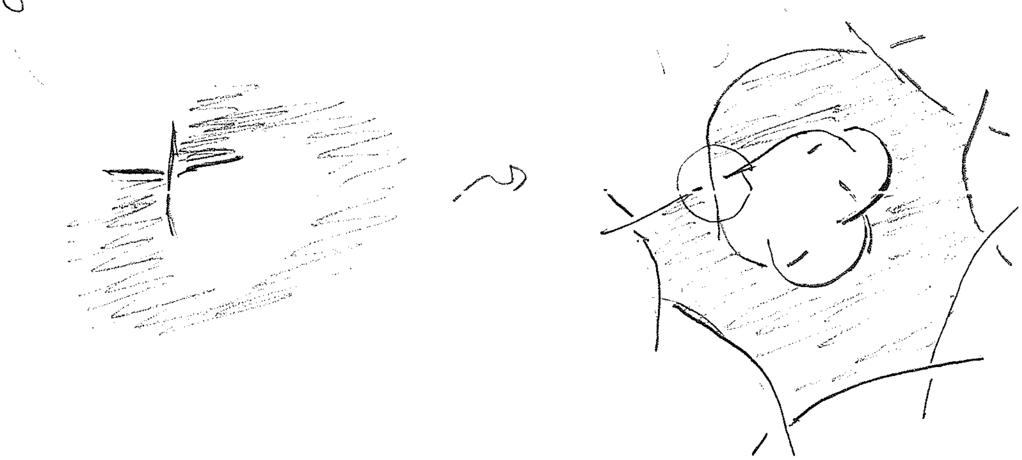


Now, checkerboard color diagram, observe that changing colors if necessary, every black region looks like this [could also have "outside" all black]

so, all A-resolution result in circles that "bound black regions".

How can switching a resolution result in more components?

region can "bump into itself",

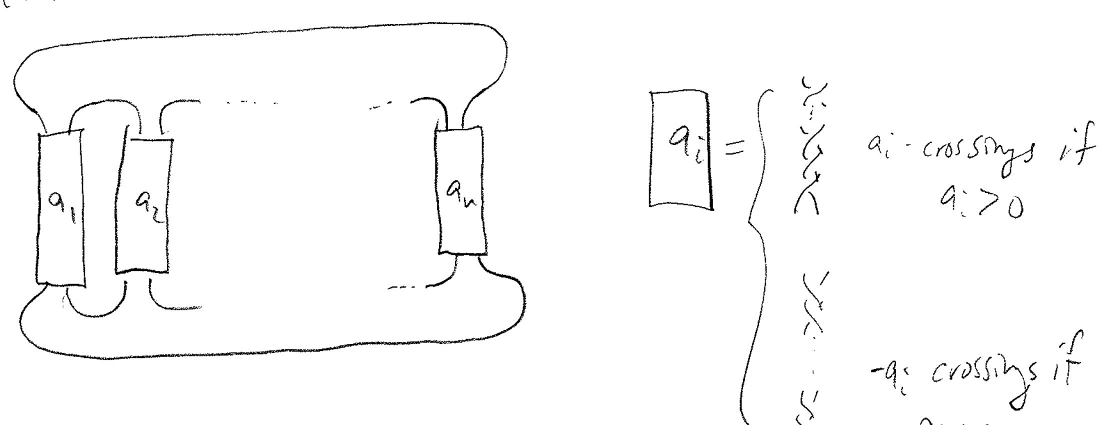


this means we have a nugatory crossing, and this contradicts assumption D is reduced. So, D is +adequate.

Similar argument show D is −adequate. □

Exercise II.16 Prove that the pretzel knot diagrams $P(p_1, \ldots, p_r, q_1, \ldots, q_s)$ are adequate if $p_i \geq 2$, $q_i \leq -2$, $r \geq 2$, $s \geq 2$

$P(a_1, \ldots, a_n)$:



Exercise II.17 For what values of $\{a_1, \ldots, a_n\}$ is this a knot?