From now on, assume $B = A^{-1}$, $d = -A^2 - A^{-2}$, so that

$\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$.

<u>Lemma II.5</u> If D, D' are related by RII, RIII moves, then $\langle D \rangle = \langle D' \rangle$.

<u>proof</u>: Exercise II.6

<u>Exercise II.7</u> Compute $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ for all diagrams of knots up to 6 crossings in the table

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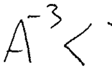
Math 428
<u>2/17/11</u>

What about RI?

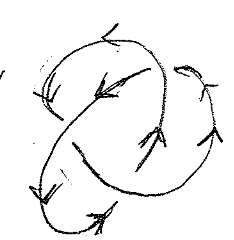
<u>Lemma II.6</u> We have $-A^{-3}\langle$  $\rangle = \langle$  $\rangle = -A^{3}\langle$  $\rangle$

<u>proof</u> Exercise II.8.

What to do? dont want to require $A^3 = -1$...

Fix requires notion of <u>writhe</u>

<u>Defn</u> If D is the diagram of an <u>oriented</u> link, the <u>writhe</u> of D is defined to be $\left(\sum_{\substack{\text{rt handed} \\ \text{crossing}}} +1 + \sum_{\substack{\text{left-handed} \\ \text{crossings}}} -1 \right) =: w(D)$.

$w\Big($  $\Big) = 3$ $\quad w\Big($ 

$\Big) = 0$

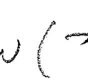
<u>Note</u> $w(D) = w(-D)$.

How does writhe change under R-moves?

R II, R III clearly do not change it.

However.

$\omega($  $) = 1 + \omega(\rightarrow)$

$\omega($  $) = -1 + \omega(\rightarrow)$

<u>Theorem II: 7</u> If D, D' are diagrams for oriented links $L \sim L'$, then $(-A)^{-3\omega(D)}\langle D\rangle = (-A)^{-3\omega(D')}\langle D'\rangle$

<u>proof</u> We can assume D & D' differ by R-move; Since $\omega(D)$ & $\langle D\rangle$ are unchanged by R II & R III moves, we can assume D & D' differ by RI move.

D:  D':  $\Rightarrow$ $\omega(D') = 1 + \omega(D)$
$\langle D\rangle = -A^{-3}\langle D'\rangle$

so

$$(-A)^{-3\omega(D)}\langle D\rangle = (-A)^{-3(\omega(D')-1)}(-A^{-3})\langle D'\rangle = (-A)^{-3\omega(D')}\langle D'\rangle$$

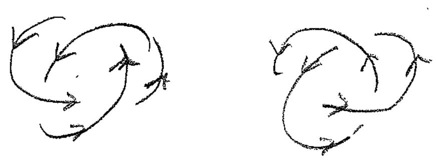
<u>Define $V_L(A) = (-A)^{-3\omega(D)}\langle D\rangle$ for any diagram D of an oriented link.</u>

<u>Corollary II. 8</u> For any oriented link $V_L(A)$ is an invariant in $\mathbb{Z}[A, A^{-1}]$ with trivial equivalence. If K is a knot, $V_K(A)$ is an invariant of unoriented knots (independent of choice of orientation used to compute).

<u>Defn</u> The <u>reflection</u> of a link L is the link $\bar{L}$ obtained by applying a reflection of $\mathbb{R}^3$ — for a diagram, just change all crossings.

Exercise II.9: Prove that $\langle D\rangle(A) = \langle \bar{D}\rangle(A^{-1})$ and $V_L(A) = V_{\bar{L}}(A^{-1})$, where D is a diagram of L and $\bar{D}$ " " " " $\bar{L}$.

Exercise II.10 Prove that the right handed & left handed trefoils are not equivalent



left right

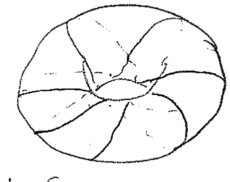
If $K \not\sim \bar{K}$, then K is called chiral. If $K \sim \bar{K}$, then K is called achiral, or amphichiral.

Exercise II.11: Prove that figure-8 is achiral .

Exercise II.12 Compute $V_{T_{2,n}}(A)$ where $T_{2,n}$ is the right-handed $(2,n)$-torus link, $\forall\, n$.

n-crossing

Can be embedded on standard torus in $\mathbb{R}^3$



goes twice around one direction, n times around the other.

[(2,3) torus knot is trefoil]

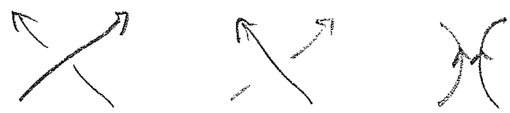
Exercise II.13 Prove that $V_L(A)$ is in $\mathbb{Z}[A^2, A^{-2}]$.

Convention: substitute $t^{1/2}$ for A^{-2} making $V_L \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$. (classical and not always agreed upon) We will see that if L is a knot, then $V_L \in \mathbb{Z}[t, t^{-1}]$ i.e. no ½-powers actually exist.

<u>Proposition II.9</u> The Jones polynomial $V_L(t) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ is an invariant of oriented links satisfying.

(i) $V_{\bigcirc}(t) = 1$, where $\bigcirc$ is oriented unknot.

(ii) Given 3 links L_+, L_-, L which are the same outside a ball, and inside satisfy



L_+ $\quad$ L_- $\quad$ L

then
$$t^{-1} V_{L_+}(t) - t V_{L_-}(t) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}}) V_L(t) = 0.$$

<u>Proof</u> 1st check behavior of bracket. (we let L denote link & diagram)

$$\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$$

$$\langle \times \rangle = A^{-1} \langle)(\rangle + A \langle \asymp \rangle.$$

$$\Rightarrow \quad A \langle \times \rangle - A^{-1} \langle \times \rangle = (A^2 - A^{-2}) \langle)(\rangle \iff$$

$$A\langle L_+ \rangle - A^{-1} \langle L_- \rangle = (A^2 - A^{-2}) \langle L \rangle$$

observe that writhes are related by
$$w(L_\pm) = w(L) \pm 1$$

so
$$(A^2 - A^{-2}) V_L(A) = (A^2 - A^{-2})(-A)^{-3w(L)} \langle L \rangle$$
$$= (-A)^{-3w(L)} (A \langle L_+ \rangle - A^{-1} \langle L_- \rangle)$$
$$= A(-A)^{-3(w(L)+1)} (-A)^3 \langle L_+ \rangle - A^{-1}(-A)^{-3(w(L)-1)} (-A)^{-3} \langle L_- \rangle$$
$$= -A^4 V_{L_+}(A) + A^{-4} V_{L_-}(A). \quad \text{substitute } t^{\frac{1}{2}} = A^{-2} \ \square$$

<u>Proposition II.10</u> If L is an oriented link with an odd number of components, then $V_L(t) \in \mathbb{Z}[t, t^{-1}]$.
If L has an even number of components, then $V_L(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ and has <u>only</u> ½-integer powers.

<u>Proof</u> induction on # of crossings in a diagram. (again, conduct diagram & link)

base case: 0 crossings, then if L has k compnts, no crossings

$w(L) = 0$ and

$$V_L(A) = \langle L \rangle = (-A^2 - A^{-2})^{k-1} = (-1)^{k-1} A^{2(k-1)} (1 + A^{-4})^{k-1}$$

substituting $t^{1/2} = A^{-2}$ we get

$$V_L(t) = (-1)^{k-1} \underbrace{t^{-\frac{(k-1)}{2}}}_{\substack{\text{introduces} \\ \text{all } \frac{1}{2}\text{-int. powers iff } k \text{ even}}} \underbrace{(1+t)^{k-1}}_{\text{poly in } t} \checkmark$$

Suppose true for up to n crossings, prove for $n+1$: —

(*) Observe that by changing some of the crossings $\times \longleftrightarrow \times$ we can make any diagram into the diagram of unlink. $\circ\circ\circ\circ$.

To go from n to $n+1$ crossings, we induct on the number of crossing changes needed to change the diagram into the diagram of unlink.

(*) - compare with example at end of 1st day.

base case here is that we require 0 crossing changes. then because V_L is a link inv't, result follows from inductive hypothesis of initial induction. Suppose we know its true now for diagrams with at most $n+1$ crossings requiring at most k crossing changes to make the diagram that of the unlink. Let L_+ be a diagram with $n+1$ crossings requiring $k+1$ crossing changes to make L_+ into the unlink. Pick one of the $k+1$ crossings and let L_- be the link obtained by changing that crossing. Then L_- requires k crossing changes to make it into the unlink, hence the result holds for L_-. Now by previous proposition we have

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_L(t) = 0$$

or

$$V_{L_+}(t) = t^2 V_{L_-}(t) + (t^{\frac{3}{2}} - t^{\frac{1}{2}})V_L(t)$$

$$= t^2 V_{L_-}(t) + t^{\frac{1}{2}}(t-1)V_L(t)$$

Further note that $|L_+| = |L_-| = |L| \pm 1$ and # of crossings of L is n, so result holds for L. $\therefore$ result holds for L_+ — to see this, for example suppose $|L_+|$ is odd, then so is $|L_-|$, while $|L_+|$ is even. From this, note that $V_L(t)$ involves only $\frac{1}{2}$-integer powers of t and hence

$t^{1/2}(t-1)V_L(t)$ involves only integer powers of t.

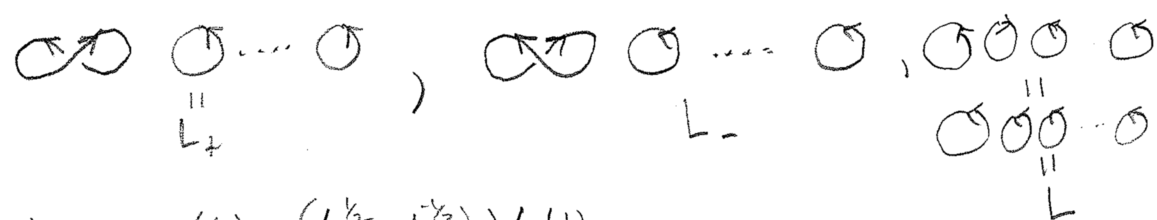
Also, $t^2 V_{L_-}(t)$ involves only integer powers of t, since $V_{L_-}(t)$ does,

so $V_{L_+}(t)$ involves only integer powers of t, as required. □.

The proof of this proposition also proves

Proposition II.11. $V_L(t)$ is determined by the two properties listed in Prop. II.10, and the fact it is an oriented link invariant.

Proof sketch. 1st check that $V_L(t) = (-t^{1/2} - t^{-1/2})^{k-1}$ for the k-component unlink L follows from properties in Prop II.10: induct on k:



$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})\, V_L(t)$$

$$\frac{t^{-1}-t}{t^{1/2}-t^{-1/2}}\, V_{L_+}(t) = V_L(t)$$

$L_+ = L_- = (k-1)$-compt unlink

$L = k$-compt unlink

$$\| $$

$$\frac{t^{-\frac{1}{2}}(1-t^2)}{t-1}\,(-t^{1/2} - t^{-1/2})^{k-2}$$

$$\|$$

$$(-t^{1/2} - t^{-1/2})(-t^{1/2} - t^{-1/2})^{k-2} \quad \checkmark$$

This is the base case for induction on # of crossings, as before it's a double induction, with 2nd induction on # of crossings required to change into unlink. □