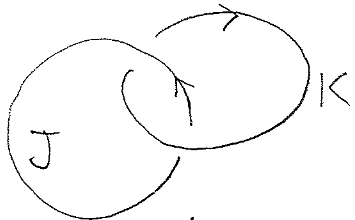
The Alexander polynomial is pretty good at distinguishing knots. Since $|A_K(-1)| = \det(K)$, we see that $A_K(t)$ determines colorability & determinant, so, in some sense, it is stronger than any invts discussed so far. [not clear for $\text{rank}_p(K)$ if > 0].

However, there is a large class of knots indistinguishable from unknot via Alexander polynomial. — Examples require a digression.

<u>Linking numbers</u>: Let L be an oriented link with two components

$L = J \cup K$

L



H = <u>Hopf link</u>

W = Whitehead link

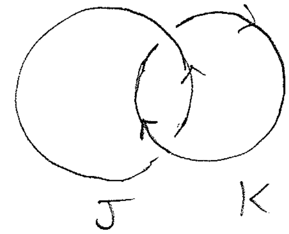
Define the <u>linking number</u> of J & K to be

$$lk(J,K) = \sum_{J \text{ under } K} \left[\begin{cases} +1 & \text{if } \overset{\downarrow K}{\longrightarrow} J \\ -1 & \text{if } \overset{\uparrow K}{\longrightarrow} J \end{cases} \right]$$

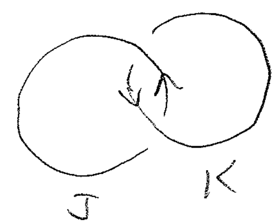
that is, pick a diagram, and add up, over all crossings of the diagram in which J passes below K, a $+1$ or -1 depending on the local picture.

Exercise II.2 Check that $lk(J,K)$ does not depend on the choice of diagram, and is unchanged by equivalence (respecting the orientations & labeling J & K), by examining the effect of Reidemeister moves.

Ex

H


$lk(J,K) = -1$
$lk(K,J) = -1$



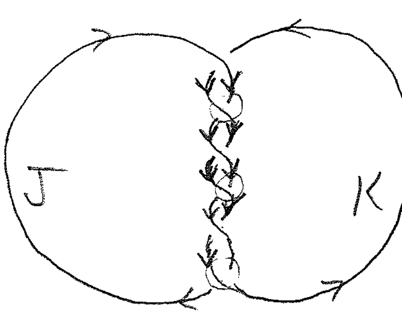
$lk(J,K) = lk(K,J) = 1$

U


$lk(J,K) = lk(K,J) = 0$

W

$lk(J,K) = lk(K,J) = 0$



L (2,6)-torus link

$lk(J,K) = lk(K,J) = -3$

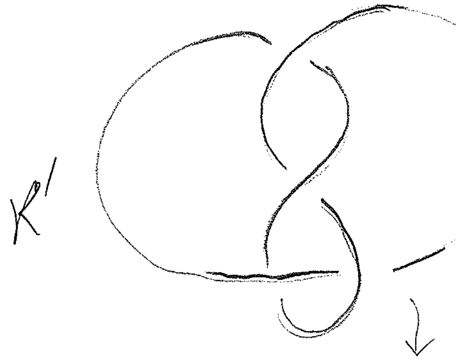
So, these are all inequivalent links.
(recall $W \not\simeq U$ via 3-colorings)

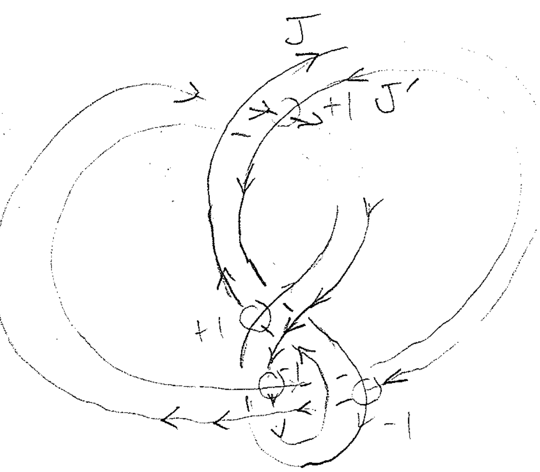
Exercise II.3 Prove that $lk(J,K) = lk(K,J) = lk(-J,K)$
$= lk(J,-K)$
for any oriented link $L = J \cup K$. (here $-J = J$ w/ reversed orientation)

Knots with trivial Alexander polynomial:

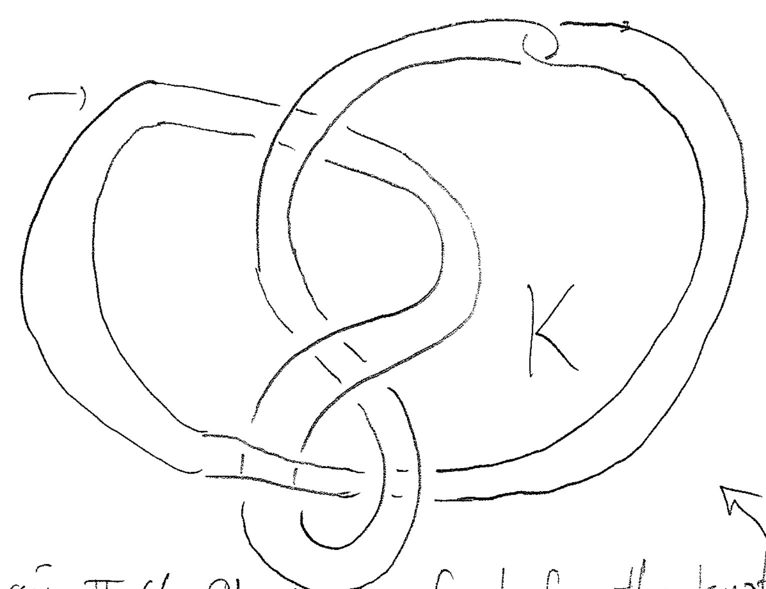
untwisted Whitehead doubles: ① Let K' be any knot.

② take a two comp't link $L = J \cup J'$ which "follow along K'" in opposite directions and which have $lk(J, J') = 0$





③ replace parallel arcs in some location by a 'clasp', to get K



K is called the untwisted Whitehead double of K'.

Fact: $A_K(t) = 1$. Exercise II.4 Check this fact for the knot shown.

Exercise II.5 Draw a diagram for two other untwisted Whitehead doubles of knots with at least 3 crossings.

Another polynomial inv't of knots & links arrived 55 years after Alexander polynomial is the <u>Jones polynomial</u> [named after V. Jones]. We'll describe a simple construction of this due to Kauffman via another polynomial called the <u>Kauffman bracket polynomial</u>. [see Lickorish Ch 3]

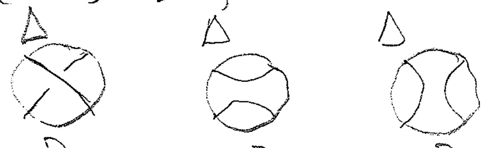
<u>Defn</u> Let D be a diagram of an unoriented link. We begin by defining a polynomial in 3-variables $\langle D\rangle \in \mathbb{Z}[A,B,d]$ by the following rules;

(i) $\langle \bigcirc \rangle = 1$: $\bigcirc$ = unknot, 1 is constant poly $=1$.

(ii) $\langle \bigcirc \cup D' \rangle = d\langle D'\rangle$: $\bigcirc \cup D'$ means the diagram D' with a disjoint unknotted comp't add.

(iii) $\langle \times \rangle = A\langle \asymp \rangle + B\langle)(\rangle$:

this means that diagrams D, D_A, D_B which differ only on some disk Δ, so that in Δ we see



D $\quad D_A \quad D_B$

D_A, D_B are said to be obtained from D by <u>resolving the crossing</u>

then
$$\langle D\rangle = A\langle D_A\rangle + B\langle D_B\rangle.$$

Observe: rules (i) & (ii) uniquely determine $\langle D\rangle = d^{k-1}$, if D is a disjoint union of k circles. Moreover, we have the following.

<u>Lemma II.2</u> Rules (i)–(iii) uniquely determine a polynomial $\langle D\rangle \in \mathbb{Z}[A,B,d]$ for any unoriented diagram D.

<u>proof</u>: The observation above is the base case of an inductive recursive proof, inducting on the # of crossings. Specifically if D is a diagram, any crossing c can be resolved to produce 2 diagrams D_A, D_B with

D, D_A, D_B

then we define

$$\langle D\rangle = A\langle D_A\rangle + B\langle D_B\rangle$$

We must prove ~~that~~ resolving any other crossing will result in same polynomial. So, suppose we resolve some other crossing c' to obtain diagrams $D_{A'}, D_{B'}$, and we need to check

$$A\langle D_A\rangle + B\langle D_B\rangle = A\langle D_{A'}\rangle + B\langle D_{B'}\rangle$$

Note that $D_A, D_B, D_{A'}, D_{B'}$ have one fewer crossings than D, so $\langle\ \rangle$ of each is uniquely determined by (i)–(iii). Further observe that D_A, D_B contain the crossing c' and $D_{A'}, D_{B'}$ contain the crossing c.

The 2 resolutions of c' on each of D_A and D_B give $D_{AA'}, D_{AB'}, D_{BA'}, D_{BB'}$ and these are the same as those obtained by resolving c in $D_{A'}, D_{B'}$.

We have

$$A\langle D_A\rangle + B\langle D_B\rangle = A\left(A\langle D_{AA'}\rangle + B\langle D_{AB'}\rangle\right) + B\left(A\langle D_{BA'}\rangle + B\langle D_{BB'}\rangle\right)$$

$$= A^2\langle D_{AA'}\rangle + AB\left(\langle D_{AB'}\rangle + \langle D_{BA'}\rangle\right) + B\langle D_{BB'}\rangle$$

and

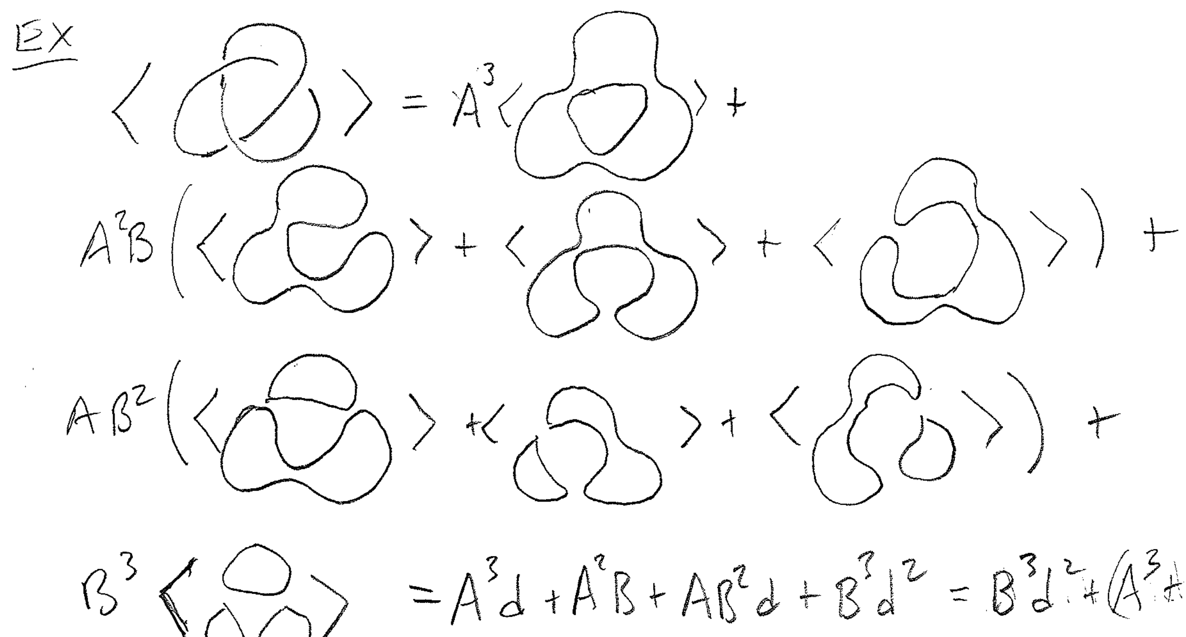
$$A\langle D_{A'}\rangle + B\langle D_{B'}\rangle = A\left(A\langle D_{AA'}\rangle + B\langle D_{BA'}\rangle\right) + B\left(A\langle D_{AB'}\rangle + B\langle D_{BB'}\rangle\right)$$
$$= A^2\langle D_{AA'}\rangle + AB\left(\langle D_{BA'}\rangle + \langle D_{AB'}\rangle\right) + B^2\langle D_{BB'}\rangle$$
$$= A\langle D_A\rangle + B\langle D_B\rangle \qquad \square$$

A choice of resolution of each crossing is called a <u>state</u> of the diagram D. If S is a state, then we write
$\langle D | S\rangle = A^i B^j$ where S has i A-resolutions and j B-resolutions (so $i+j=k$). We also write $|S| = \#$ of circles in resolution of D determined by S. The previous proof not only gives well definedness of $\langle D\rangle$, but in fact we have

<u>Proposition II.3</u> If D is a diagram of an unoriented link then

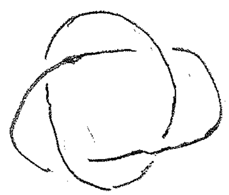
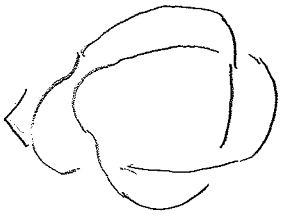
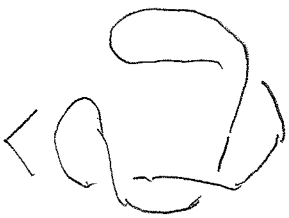
$$\langle D\rangle = \sum_S \langle D|S\rangle d^{|S|-1}.$$

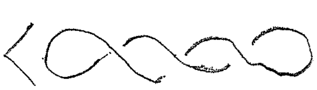
<u>EX</u>



$$= A^3 d + A^2 B + AB^2 d + B^3 d^2 = B^3 d^2 + (A^3 + AB^2) d + A^2 B$$

Can build up tables since previous computations can be used to compute new examples:

Ex $\langle$  $\rangle = A \langle$  $\rangle + B \langle$  $\rangle$

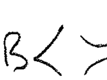
$= A\left(A^2B + (A^3 + AB^2)d + B^3d^2\right) + B\langle$  $\rangle$

$= \underbrace{\qquad\qquad\qquad\qquad}_{\text{"}} + B\left(A^3 + A^2B(d+d+d) + AB^2(d^2+d^2+d^2) + B^3d^3\right)$

$= A^3B + A^4d + A^2B^2d + B^3d^2 + A^3B + 3A^2B^2d + 3AB^3d^2 + B^4d^3$

Not an inv't, but use R-moves to decide what is needed.

Lemma II.4 For any diagram we have

$\langle$  $\rangle = AB\langle$  $\rangle + (ABd + A^2 + B^2)\langle\ )(\ \rangle$

In particular, $\langle\ \rangle$ is invariant under R II if
$AB = 1$ and $d = -A^2 - A^{-2}$.

proof: $\langle$  $\rangle = A^2\langle$  $\rangle + AB\left(\langle\ )\circ(\ \rangle + \langle\right.$  $\left.\rangle\right)$
$+ B^2\langle$  $\rangle$

$= A^2\langle\ )(\ \rangle + ABd\langle\ )(\ \rangle + AB\langle \asymp \rangle + B^2\langle\ )(\ \rangle$

$= (A^2 + ABd + B^2)\langle\ )(\ \rangle + AB\langle \asymp \rangle$ ✓

so with eqns in A, B, d get

$\langle \asymp \rangle = (A^2 + 1(-A^2 - A^{-2}) + A^{-2})\langle\ )(\ \rangle + \langle \asymp \rangle = \langle \asymp \rangle$ $\square$

From now on, assume $B = A^{-1}$, $d = -A^2 - A^{-2}$, so that

$\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$.

Lemma II.5 If D, D' are related by RII, RIII moves, then $\langle D \rangle = \langle D' \rangle$.

proof: Exercise II.6

Exercise II.7 Compute $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ for all diagrams of knots up to 6 crossings in the table

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Math 428
2/17/11

What about RI?

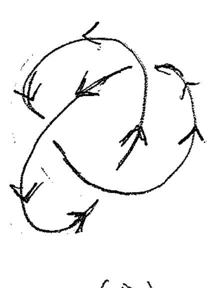
Lemma II.6 We have $-A^{-3}\langle\ \rangle = \langle\ \rangle = -A^{3}\langle\ \rangle$

proof Exercise II.8.

What to do? dont want to require $A^3 = -1$...

Fix requires notion of writhe

Defn If D is the diagram of an oriented link, the writhe of D is defined to be $\left(\sum_{\substack{\text{rt handed} \\ \text{crossing}}} +1 + \sum_{\substack{\text{left-handed} \\ \text{crossings}}} -1 \right) =: w(D)$.

$w($  $) = 3$ $w($  $) = 0$

Note $w(D) = w(-D)$.