II. Polynomial Invariants

Alexander polynomial (1st look)

There is a refinement of determinant –

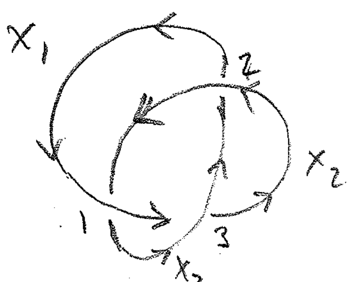
Alexander polynomial of $K = A_K(t) \in \mathbb{Z}[t, t^{-1}]$ = Laurent polys over $\mathbb{Z}$.

so $A_K(t) = \sum_{j=-N}^{N} a_j t^j$

We can think of $\mathbb{Z}[t, t^{-1}] = \mathbb{Z}[G]$, where $G = \langle t \rangle \cong \mathbb{Z}$
integral group ring

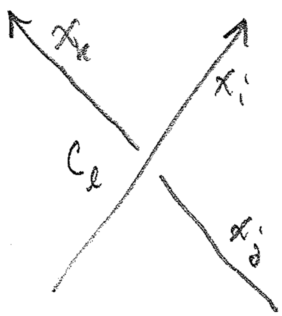
Defined as follows:

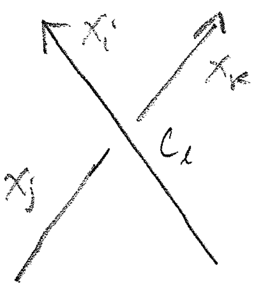
1st orient the knot – give it a direction
(2 choices for a knot, 2^k for a k-compt link)



$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1-t & t & -1 \\ -1 & 1-t & t \\ t & -1 & 1-t \end{pmatrix}$$

Construct a matrix with rows for crossings, columns for overarcs, and entries as follows





If $x_i = x_j$, $x_j = x_k$, or $x_i = x_k$ (or all 3 =) then add up

	x_i	x_j	x_k
c_l	$1-t$	-1	t

	x_i	x_j	x_k
c_l	$1-t$	t	-1

Get $M_D(t) \in M_{k\times k}(\mathbb{Z}[t])$, $M_D'(t)$ any minor is an Alexander matrix for D

$A_D(t) = \det(M_D'(t))$ is "the" Alexander polynomial of D.

Observe: $|A_D(-1)| = \det(D)$.

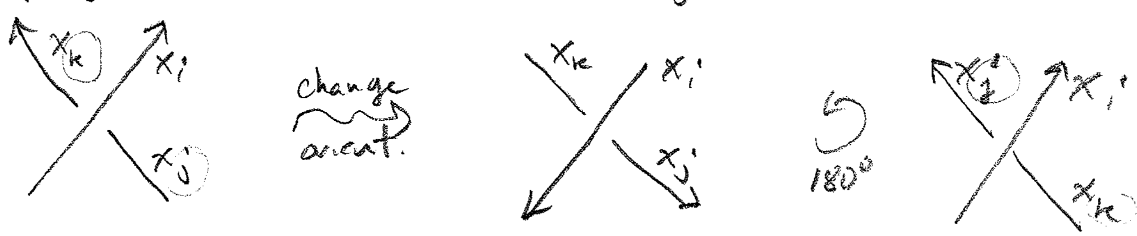
Theorem II.1 If D, D' are diagrams for knots $K \sim K'$, then

$A_D(t) = \pm t^k A_{D'}(t)$ for some $k \in \mathbb{Z}$.

In fact, $A_D(t)$ depends on choice of minor, up to $\pm t^k$ factor.

proof of this, and that $A_D(t) = \pm t^k A_{D'}(t)$ for _oriented_ diagrams is similar to, but more involved than, invariance of det.

Changing orientation: replace t by t^{-1} (up to $\pm t^k$):



i	j	k
$1-t$	-1	t

i	j	k
$1-t$	t	-1

$$1-t,\ -1,\ t \xmapsto{t \mapsto t^{-1}} 1-t^{-1},\ -1,\ t^{-1} \xmapsto{\cdot(-t)} -t+1,\ t,\ -1.$$

so, change orientation $D \rightsquigarrow -D$ get $A_D(t) = \pm t^k A_{-D}(t^{-1})$

Fact: up to $\pm t^k$, $A_D(t)$ is _palindromic_: $A_D(t) = \pm t^k A_D(t^{-1})$.

Exercise II.1 Compute $A_K(t)$ for each knot K w/ less than 6 crossings in the table.