Math 428
1-27-11

Last time: Given a prime $p \geq 3$, we defined

$C_p(K)$ = vectorspace over $\mathbb{Z}/p\mathbb{Z}$ of p-colorings of a diagram of K.

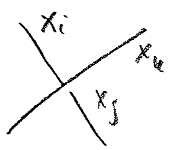
$\dim C_p(K) \geq 1$ (trivial colorings — all the same value)

Define the <u>mod p rank</u> of $K = \mathrm{rank}_p(K) = \dim C_p(K) - 1$

<u>Exercise I.7</u> # of nontrivial p-colorings of $K = (\mathrm{rank}_p(K) \cdot p - 1)p$

Also computed $\mathrm{rank}_p(K)$ from a matrix as follows

D = diagram for K with k crossings



$2x_k = x_i + x_j \pmod p$.

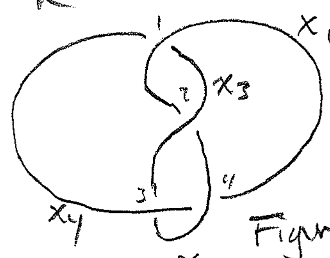


Figure 8 Knot

$$\Rightarrow \quad M_D = \begin{pmatrix} 2 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & -1 & 2 \\ -1 & 2 & 0 & -1 \end{pmatrix}$$

Let M_D' be some $(k-1)\times(k-1)$ minor (any will do) and thought of as a matrix with entries in $\mathbb{Z}/p\mathbb{Z}$.

then $\mathrm{rank}_p(K) = \dim(\mathrm{null}(M_D'))$,

$$M_D' = \begin{pmatrix} 2 & 0 & -1 \\ -1 & -1 & 2 \\ 0 & -1 & -1 \end{pmatrix}$$

in particular $\mathrm{rank}_p(K) \neq 0$ iff $p \mid \det(M_D')$

$$\det(M_D') = 2(3) - 1(1) = 6 - 1 = 5$$

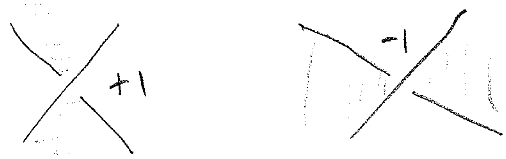
In Figure 8, $\mathrm{rank}_p(K) = 0$ unless $p = 5$.

then, compute $\mathrm{rank}_5(K)$ using row reduction.

$$\begin{pmatrix} 2 & 0 & -1 \\ -1 & -1 & 2 \\ 0 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & -2 & 3 \\ -1 & -1 & 2 \\ 0 & -1 & -1 \end{pmatrix} \underset{\text{mod } 5}{\rightsquigarrow} \begin{pmatrix} 0 & 3 & 3 \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \underset{\text{mod } 5}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so, $\mathrm{rank}_5(K) = 1$.

Idea was; multiply certain rows by -1 according to black/white checker-board coloring



then sum of rows $= 1$.

<u>Exercise I.8</u>: If A is a $k \times k$ matrix and all rows and columns sum to 1, prove that every $(k-1)\times(k-1)$ minor has the same determinant, up to sign.

<u>Defn</u> If D is a diagram for K w/ k crossings, define <u>determinant of D</u> $= \det(D)$ to be $|\det(M_D')|$, for any $(k-1)\times(k-1)$ minor of M_D (det of $0 \times 0 \equiv 1$)

by exercise, this is independent of the choice of minor.

By Cor. I.11, the prime divisors of $\det(D)$ give an inv't of K, and in fact we have

<u>Proposition I.12</u> If D is a diagram of K, then $\det(D)$ is an inv't of K.

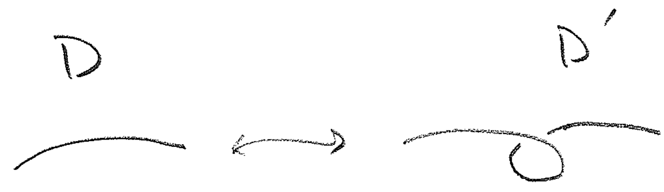
We write $\det(D) =: \det(K)$, call this the determinant of K.

<u>Exercise I.9</u> Compute determinant and mod p rank for all knots up to 6 crossings (see tables online), and all prime $p \geq 3$.

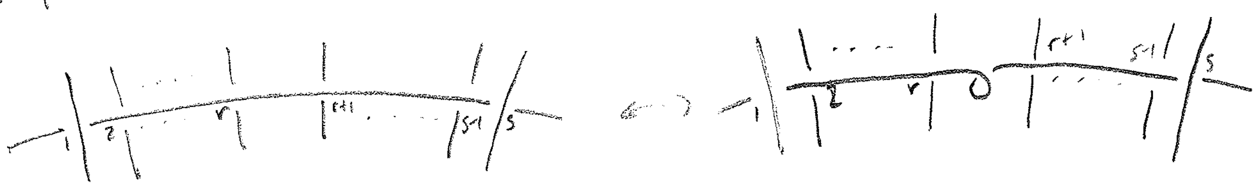
<u>proof of Proposition</u>: Need to check invariance by Reidemeister moves. [Convenient choice of numbering is helpful.].

<u>Exercise I.10</u> Prove that for <u>knots</u>, there are no nontrivial 2-colorings. <u>I.11</u> Prove $\det(K)$ is odd $\forall$ knots K.

R.I



in D label overarc involved x_1, let $c_1, \ldots, c_r, c_{r+1}, \ldots, c_s$ be crossings x_1 passes through, in order, so that $c_1, \ldots, c_r$ before ∂^- $c_{r+1}, \ldots, c_s$ after



in D' x_1 becomes x_0', x_1' and $x_i = x_i'$ $\forall i > 1$ [Recall diagrams agree outside]

c_0 = new crossing, $c_i = c_i'$ $i \geq 1$

$$M_D \qquad \begin{array}{c} 1 \\ \vdots \\ \vdots \\ r \\ \\ \vdots \\ s-1 \\ s \\ \\ \\ \end{array} \begin{bmatrix} -1 & & \\ 2 & & \\ \vdots & & \\ 2 & & \\ 2 & * & \\ \vdots & & \\ 2 & & \\ -1 & & \\ 0 & & \\ 0 & & \end{bmatrix}$$

$$M_{D'} \qquad \begin{array}{c} 0 \\ 1 \\ \vdots \\ \vdots \\ r \\ \\ \vdots \\ s-1 \\ s \\ \vdots \\ \\ \end{array} \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 0 & & & \\ 2 & 0 & & & \\ \vdots & \vdots & & & \\ 2 & 0 & & * & \\ 0 & 2 & & & \\ \vdots & \vdots & & & \\ & 2 & & & \\ 0 & -1 & & & \\ \vdots & & & & \end{bmatrix}$$
$$\qquad\qquad\quad \begin{array}{ccc} 0 & 1 & \cdots \end{array}$$

add column 0 to column 1 in $M_{D'}$ (don't change det)

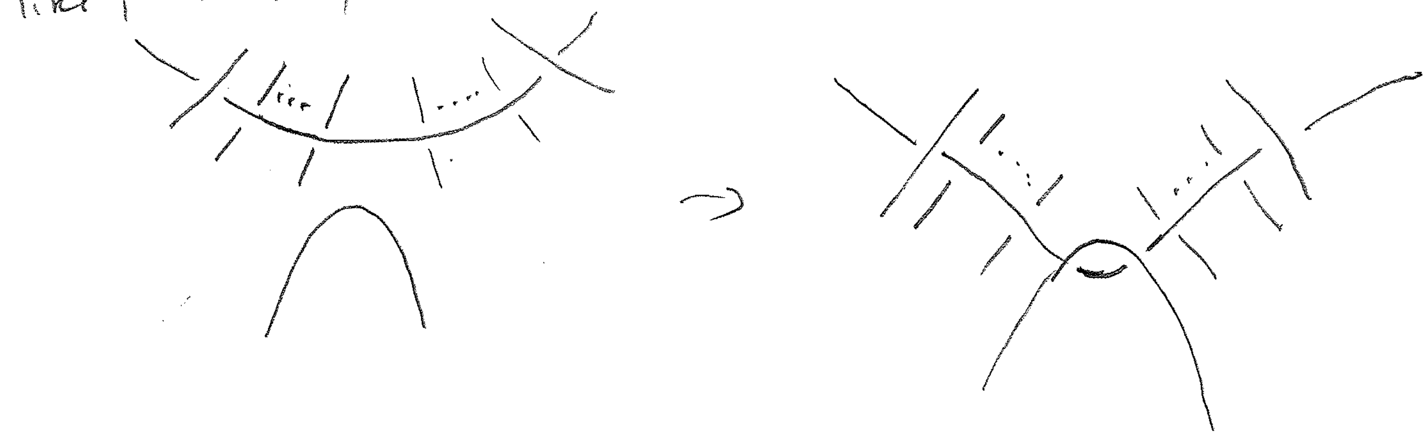
$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & & & \\ 2 & 2 & & & \\ \vdots & \vdots & & & \\ 2 & 2 & & * & \\ 0 & 2 & & & \\ \vdots & \vdots & & & \\ & 2 & & & \\ 0 & -1 & & & \\ \vdots & & & & \end{bmatrix}$$

remove last row & column, $\Rightarrow M_{D'}'$ look at cofactor expansion of $\det(M_{D'}')$ along top row — all terms are zero except $1 \cdot \det(M_D')$

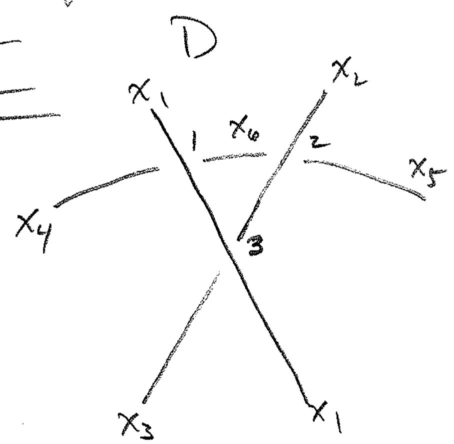
[Need to be a little careful — could have $c_s = c_j$, some j :

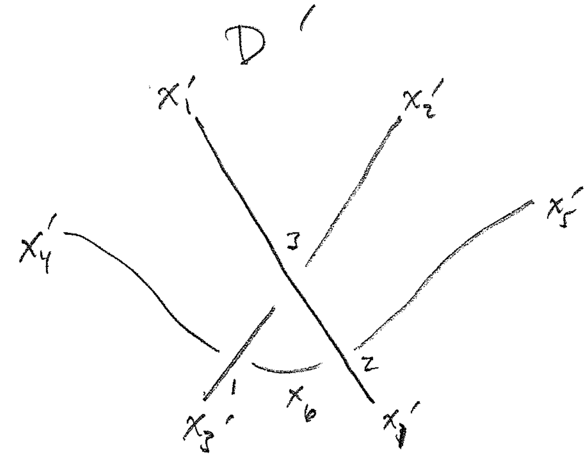
$\rightsquigarrow$ similiar argument, maybe this is all crossings,.. special care with low crossing — $\det(0 \times 0\text{-matrix}) \equiv 1$ ...]

Exercise I.12 Check RII-invariance. (assume, if you like, that picture is with 1st & last crossing distinct.



RIII





$c_i = c_i'$ & $x_i' = x_i$ outside

$$M_D = \left[\begin{array}{cccccc|ccc}
2 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \cdots & 0 \\
0 & 2 & 0 & 0 & -1 & -1 & 0 & 0 \cdots & 0 \\
2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \cdots & 0 \\
\hline
 & & & *_1 & & & 0 \vdots 0 & & *_2
\end{array}\right]
\qquad
M_{D'} = \left[\begin{array}{cccccc|ccc}
0 & 0 & 2 & -1 & 0 & -1 & 0 & 0 \cdots & 0 \\
2 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \cdots & 0 \\
2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \cdots & 0 \\
\hline
 & & & *_1 & & & 0 \vdots 0 & & *_2
\end{array}\right]$$

Pick right minors, 1st row & column, expand along new 5th column using cofactor expansion. — same matrices. □

[special cases again ...