Math 428
1-25-11



Last time –

Introduced $1^{st}$ knot & link invariant, Tri : Link $(\mathbb{R}^3) \longrightarrow \mathbb{Z}$

where Link$(\mathbb{R}^3) = \{ L \subset \mathbb{R}^3 \mid L$ a link$\} \supset$ Knot$(\mathbb{R}^3)$.

Tri$(K)$ = # of tricolorings of diagram D for K (color over arcs w/ 3 colors, all or one at crossings) at least 2 colors.


It is an invt (i.e, Tri$(K)$ = Tri$(K')$ if $K \sim K'$), by Reidemeister Thm we need only check Tri is invt by isotopy (of course) & 3 moves.

More colors? Yes, and more structure...

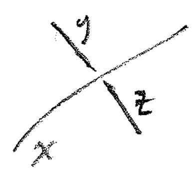
Let $\mathbb{Z}/p\mathbb{Z}$ = integers mod $p$ ($p$ a prime $\geq 3$) = quotient of $\mathbb{Z}$ by prime ideal $p\mathbb{Z}$

= equivalence classes in $\mathbb{Z}$ via equiv. rel'n $x \sim y \iff p \mid y - x$.

addition & multiplication of equiv. classes (or any representative) define a field structure on $\mathbb{Z}/p\mathbb{Z}$.

A (nontrivial) p-coloring is an assignment of elt of $\mathbb{Z}/p\mathbb{Z}$ to each over arc

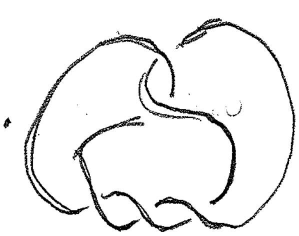
s.t.




$\Rightarrow$ $z + y = 2x$ (mod p.) (and use at least 2 colors)

Exercise I.4 A nontrivial 3-coloring in this sense is the same as tricoloring with colors $\mathbb{Z}/3\mathbb{Z}$.

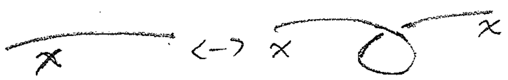
Exercise I.5. Find a nontrivial 7-coloring of the $5_2$ knot.



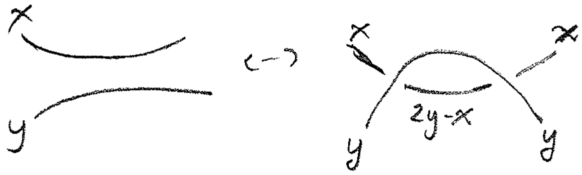
Exercise I.6 Show that there is no nontrivial p-coloring of the trefoil for $p > 3$.

**Proposition I.7** The number $col_p^{(0)}(K)$ of (nontrivial) p-colorings is a knot/link invariant.

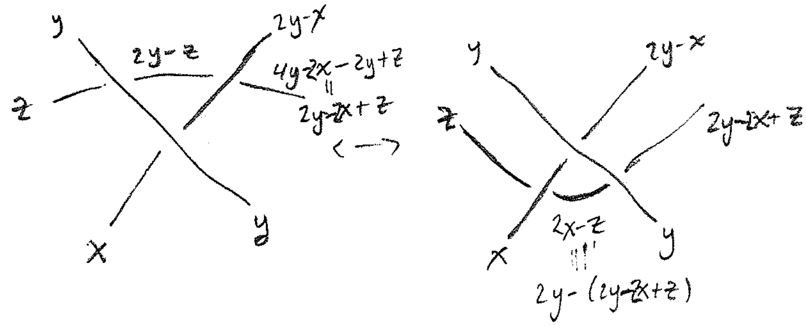
Proof:




$2x = x + x$




left is nontrivial iff right is.




$\square$

Suppose K is a knot, D a diagram. Let $X_D$ = overcrossing arcs of D.

The set of p-colorings is a subset of functions

$C_p(D) \subset \{ : X_D \to \mathbb{Z}/p\mathbb{Z}\} \leftarrow$ vector space $\cong (\mathbb{Z}/p\mathbb{Z})^{|X_D|}$

Observe that $C_p(D)$ is closed under addition & $\mathbb{Z}/p\mathbb{Z}$-scalar multiplication.

$\Rightarrow$ its a vector subspace!

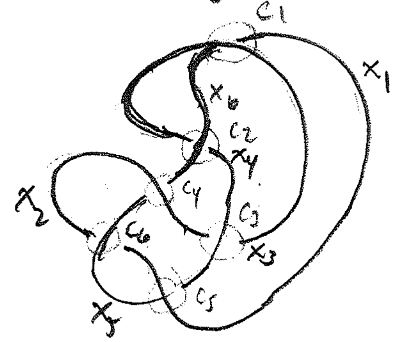
proof of Prop I.7 actually produces an isomorphism $C_p(D) \cong C_p(D')$ whenever $D \leftrightarrow D'$ by R-moves

**Corollary I.8** $C_p(K) = C_p(D)$ defines an invariant into isomorphism classes of $\mathbb{Z}/p\mathbb{Z}$-vector spaces.

How to compute? (or decide nontriviality) system of linear eqns.

Let $x_1, \dots, x_k$ variables, one for each over arc in $X_D$,

for each crossing, get a linear eqn (w/ coeffs in $\mathbb{Z}/p\mathbb{Z}$)




$c_1$: $x_1 + x_6 = 2x_3$ or $2x_3 - x_1 - x_6 = 0$

$$\begin{pmatrix} -1 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 & 2 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & -1 \\ 2 & 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \end{pmatrix} = M$$

$\rightsquigarrow$ matrix for diagram $M_D$. — can think of this in $M_k(\mathbb{Z})$ or $M_k(\mathbb{Z}/p\mathbb{Z})$ for any $p$.

a $p$-coloring is ~~a~~ exactly a vector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \in (\mathbb{Z}/p\mathbb{Z})^k$ st. $M_D x = 0 \pmod p$.

$\mathrm{null}_p(M_D) = $ null space mod $p \cong C_p(D)$

What about nontrivial colorings? $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is always a coloring (basis $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ for that matter)

Assuming last color, $x_k = 0$, we want a nontrivial soln to

$$M_D \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ 0 \end{pmatrix} = 0 \iff \hat{M}_D \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \end{pmatrix} = 0 \qquad \hat{M}_D = M_D - \text{last column}.$$

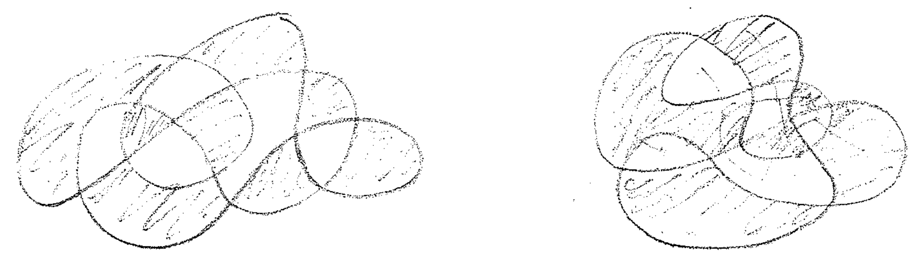
nontrivial $p$-coloring exists iff $\hat{M}_D x = 0$ has nontrivial soln mod $p$, i.e. $\mathrm{null}_p(\hat{M}_D)$ nontrivial

Want to detect this with a determinant — which one?

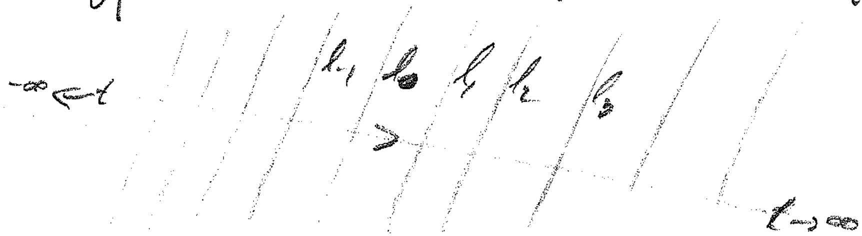
Recall: $\mathrm{null}_p(\hat{M}_D) = \mathrm{row}_p(\hat{M}_D)^\perp$.

Lemma I.9 Any row of $M_D$ (or $\hat{M}_D$) is a linear combination of others.

**Lemma I.10** $\exists$ a checkerboard coloring Black/White of complementary regions of $D$ so no two colors share an edge.

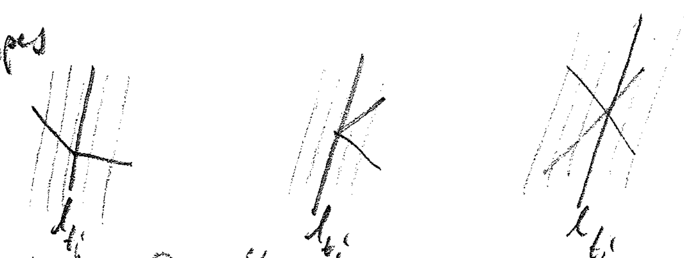


Proof: Let $\ell$ be a line not parallel to any of edges, not parallel to a line containing multiple vertices. Consider the one-parameter family of lines $\parallel$ to $\ell$ sweeping out $\mathbb{R}^2$, $\{\ell_t\}_{t\in\mathbb{R}}$

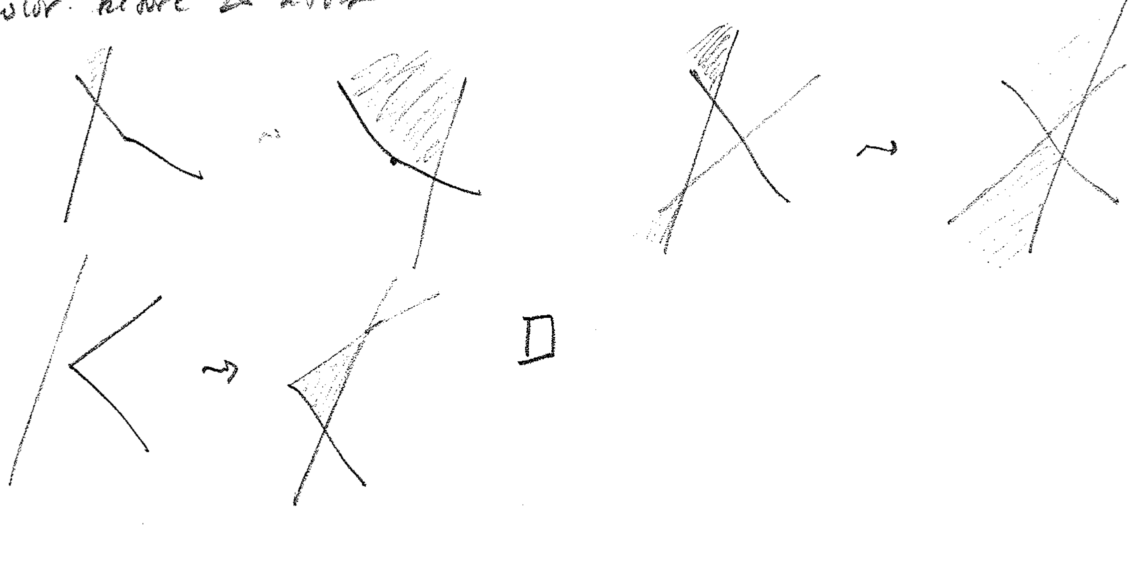



color as we sweep out $\mathbb{R}^2$: for $t << 0$, color all white.
"critical points" $t_0 < t_1 < \dots < t_n$ occur when $\ell_{t_i}$ meets a vertex.

3 types




color before & after



$\square$

proof of L I.9 rows of $M_D \longleftrightarrow$ crossings $c_1, \dots, c_k$.

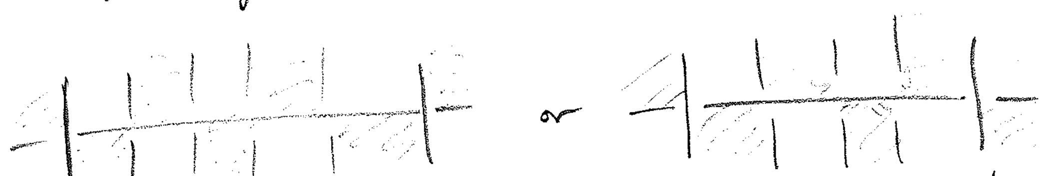
Claim: Can assign +1 or −1 to each crossing so that associated sum of ± rows is 0.

pf checkerboard color diagram. Declare signs by

+1 −1

black "right twist" black "left twist"

to check corresponding sum of ± rows = 0, suffices to check on each column — these correspond to over arcs. Following the arc along, we see



−1 +2 −2 +2 −2 +1 or −1 +2 −2 +2 −1

even odd

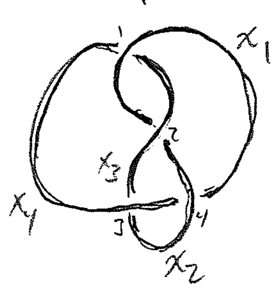
0 ✓ 0 ✓ □

Claim proves the lemma. □

Now observe that $\mathrm{row}_p(\hat{M}_D)^\perp = \mathrm{row}_p(\hat{\hat{M}}_D)^\perp$, where $\hat{\hat{M}}_D$ is obtained by removing a row from $\hat{M}_D$ (so, $\hat{\hat{M}}_D$ is a minor of $M_D$)

Corollary I.11: $\det(\hat{\hat{M}}_D) = 0 \mod p \iff \exists$ nontrivial $p$-coloring.

Ex. Figure 8




$M = \begin{pmatrix} \; \end{pmatrix}$

Defn: mod p rank of $K$ = dim of $\mathrm{null}(\hat{M}_D) = \dim(\mathrm{row}_p \hat{M}_D)$ (dim as $\mathbb{Z}/p\mathbb{Z}$ v.space)
$= \dim(C_p(K)) - 1$