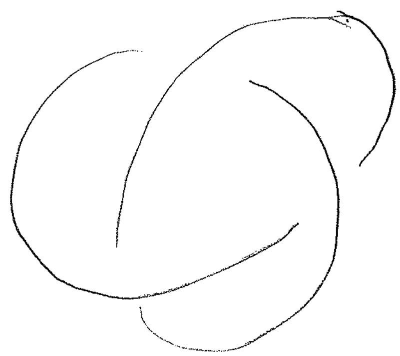
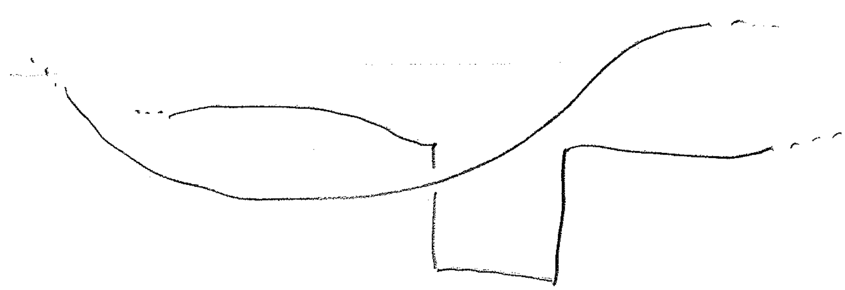
Wirtinger presentation

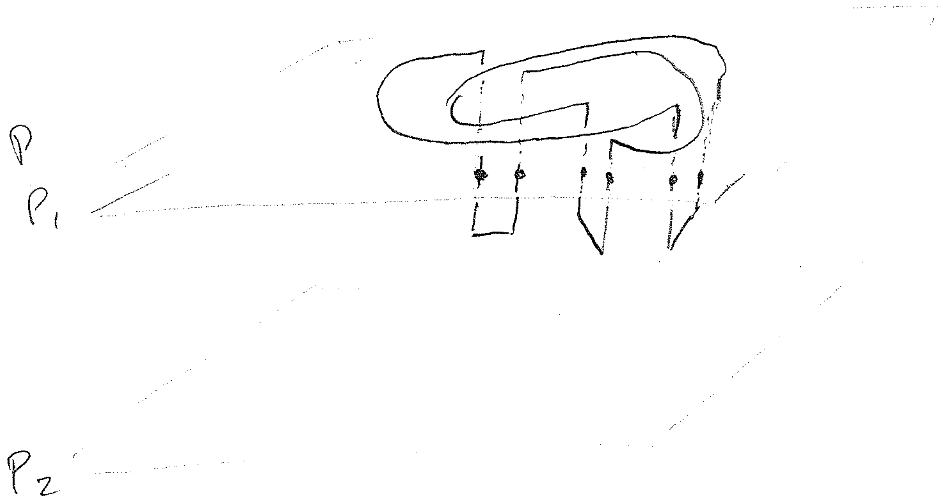
L a link in S^3 or $\mathbb{R}^3$, $X_L = S^3 \setminus N(L)$, $N(L)$ a small ε-nbhd. (or just take $S^3 \setminus L$). Look at at projection



Up to isotopy, not changing projection, we can assume L is contained in projection plane, P, except where it dips below at an under crossing as shown:

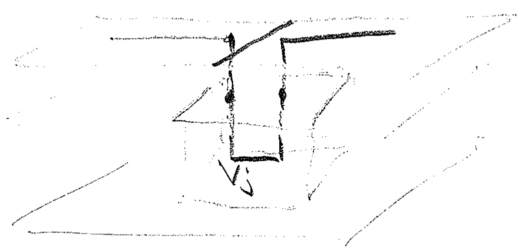


Let P_1, P_2 be planes parallel and below P so that P_1 cuts through the dip down and P_2 is disjoint from L.



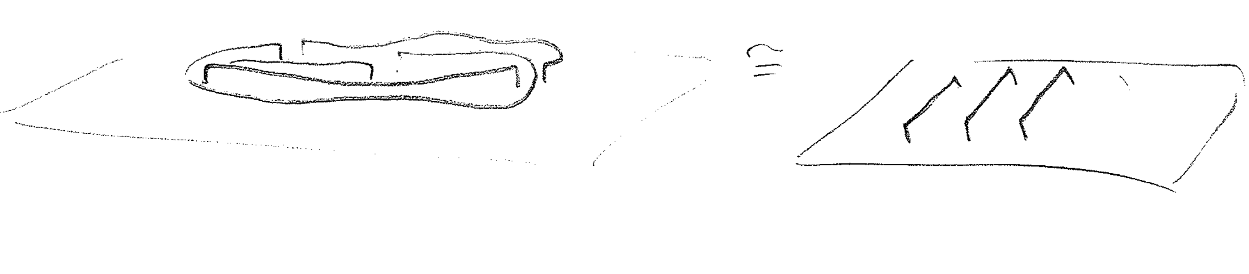
If $X_L' \subset X_L$ is the open set above P_2, it's easy to see that inclusion $i: X_L' \longrightarrow X_L$ induces an isomorphism $i_*: \pi_1(X_L') \longrightarrow \pi_1(X_L)$ (can use V.K.) So, we just compute $\pi_1(X_L)$.

If $c_1, \ldots, c_n$ are crossings, we find open sets $U, U_1, \ldots, U_n$ to which we can apply V.K. 1st we construct U: for each c_j, we remove a cube V_j between P_1 & P_2 containing the arc of L determining c_j:

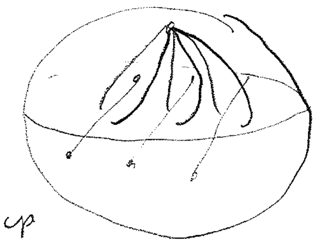


$$U = X_L' - \bigcup_{j=1}^{n} V_j$$

U looks like ½-space minus nbhds ⊥ arcs:



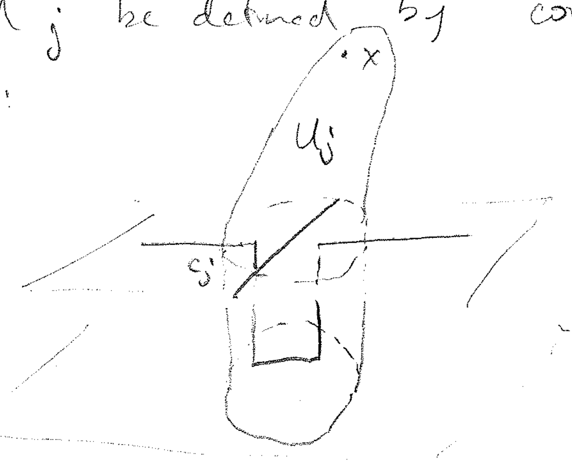
pick basepoint x high above, then

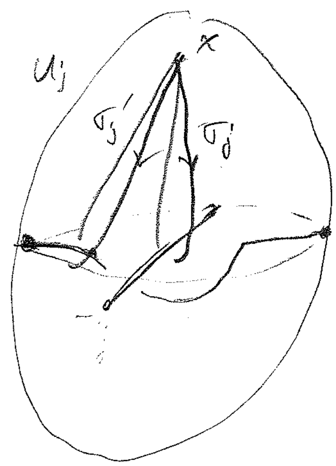
$\pi_1(U,x) \cong F_n$. $\quad U \cong$  h.e. 

gen. by loops $\gamma_1, \ldots, \gamma_n$ from x down to arc & back up

$\pi_1(U,x) = \langle \gamma_1, \ldots, \gamma_n \mid - \rangle$

Let U_j be defined by "coning" to x, and enlarging slightly:





$\pi_1(U_j, x) = \langle \sigma_j, \sigma_j' \mid - \rangle$

V.K. Thm $\Rightarrow$

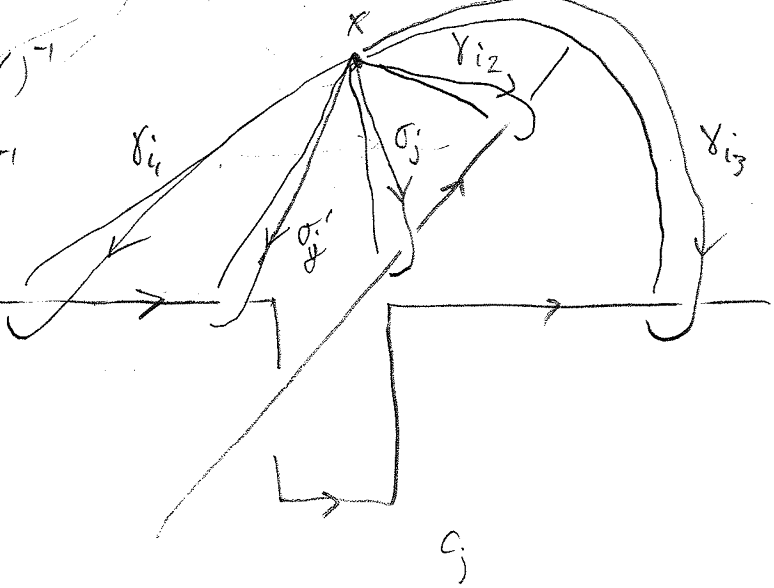
$\pi_1(X_2, x) \cong \pi_1(X_2', x) = \langle \gamma_1, \ldots, \gamma_n, \sigma_1, \sigma_1', \ldots, \sigma_n, \sigma_n' \mid r_1', r_1'', r_1''', \ldots, r_n', r_n'', r_n''' \rangle$

where r_j, r_j', r_j'' are defined as follows:

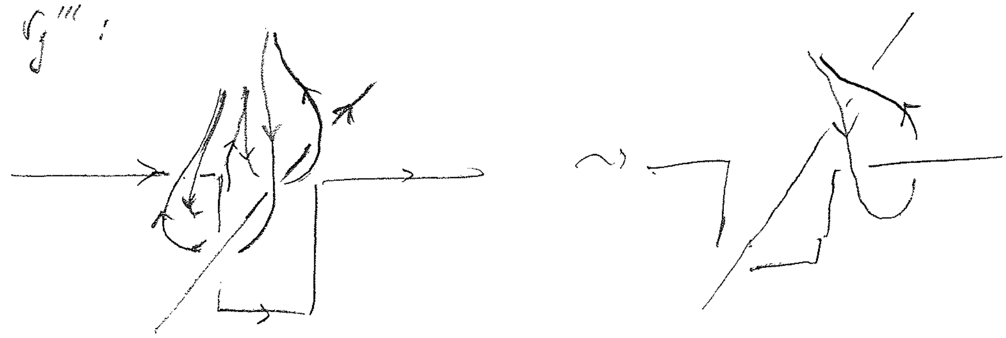
$r_j' = \gamma_{i_1}(\sigma_j')^{-1}$

$r_j'' = \gamma_{i_2}(\sigma_j)^{-1}$

$r_j''' = \sigma_j \sigma_j' \sigma_j^{-1} (\gamma_{i_3})^{-1}$



To see r_j''':



Observe that r_j' & r_j'' say that in $\pi_1(X_L', x)$, $\sigma_j' = \gamma_{i_1}$ and $\sigma_j = \gamma_{i_2}$.

So, substituting into r_j'' we get

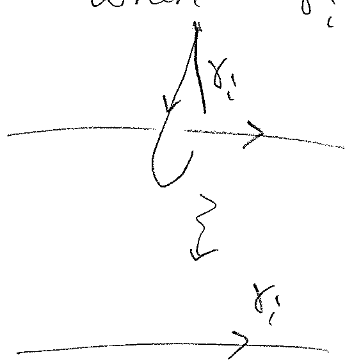
$$r_j = \gamma_{i_2}\gamma_{i_1}\gamma_{i_2}^{-1}\gamma_{i_3}$$

This gives the Wirtinger presentation:

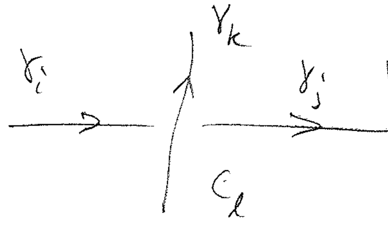
Thrm III.100

$$\pi_1(X_L, x) = \langle \gamma_1, \dots, \gamma_n \mid r_1, \dots, r_n \rangle$$

where γ_i corresponds to an over crossing arc

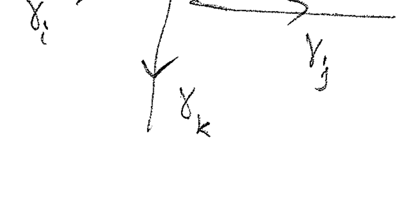


and at ℓ^{th}-crossing



$r_\ell = \gamma_k \gamma_i \gamma_k^{-1} \gamma_j^{-1}$

or

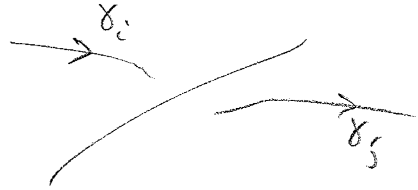


$r_\ell = \gamma_k^{-1} \gamma_i \gamma_k \gamma_j^{-1}$.

Corollary III.101 If L is a link, then $\pi_1(X_L)^{ab} \cong \mathbb{Z}^{|L|}$, where $|L| = \#$ of components.

<u>proof</u>: If we add commuting relations $[\gamma_i, \gamma_j] = 1$, then the relations $r_k = \gamma_k \gamma_i \gamma_k^{-1} \gamma_j^{-1}$ or $\gamma_k^{-1} \gamma_i \gamma_k \gamma_j^{-1}$ become $\gamma_i \gamma_j^{-1}$

so, in $\pi_1(X_L)^{ab}$, $\gamma_i = \gamma_j$ when γ_i, γ_j are arcs of same compt of L.



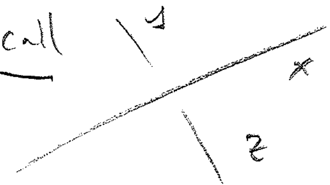
So, if $\gamma_1, \dots, \gamma_{|L|}$ are arcs, one for each compt of L, then

$$\pi_1(X_L)^{ab} = \langle \gamma_1, \dots, \gamma_{|L|} \mid \{[\gamma_i, \gamma_j] = 1\}_{i,j} \rangle \cong \mathbb{Z}^{|L|}. \square$$

Corollary III.102 Let K be a knot. $\exists$ a bijection between the sets:

$$\{\text{nontrivial mod } p \text{ colorings}\} \longleftrightarrow \{\varphi : \pi_1(X_K) \to D_{2p} \mid \varphi \text{ a surjective hom}\}$$

proof Recall



$x, y, z \in \mathbb{Z}/p\mathbb{Z}$,

$2x \equiv y + z \mod p$

$D_{2p} = \{1, s, s^2, \dots, s^{p-1}, t, ts, \dots, ts^{p-1}\}$

t = reflection in x-axis.

s = rotation by angle $\frac{2\pi}{p}$.

$ts^x = s^{-x} t$

observe all generators of $\pi_1(X_L)$ in Wirtinger presentation are conjugate.

Since rotations don't generate, if $\varphi : \pi_1(X_K) \to D_{2p}$ is onto must have $\varphi(\gamma_i) = ts^{x_i}$ $\forall i = 1, \dots, n$.

A set map $\varphi : \{\gamma_1, \dots, \gamma_n\} \to \{t, ts, \dots, ts^{p-1}\}$ extends to a hom. if all relations satisfied:

set $\varphi(\gamma_i) = ts^{x_i}$, the relation $\gamma_k^{-1} \gamma_i \gamma_k \gamma_j^{-1} = 1$ gives

$$1 = (ts^{x_k})^{-1}(ts^{x_i})(ts^{x_k})(ts^{x_j})^{-1} = s^{-x_k} t^2 s^{x_i} s^{-x_k} t^2 s^{x_j}$$
$$= s^{x_i + x_j - 2x_k} \iff x_i + x_j \equiv 2x_k \mod p \ \square$$

Alexander polynomial revisited

Let K be a knot, Let $S \subset X_K$ be a Seifert surface (or a Seifert surface intersected with X_K).

by Corollary III.101, $\pi_1(X_K)^{ab} \cong \mathbb{Z}$. Let

$$\varphi_K : \pi_1(X_K) \longrightarrow \mathbb{Z}$$

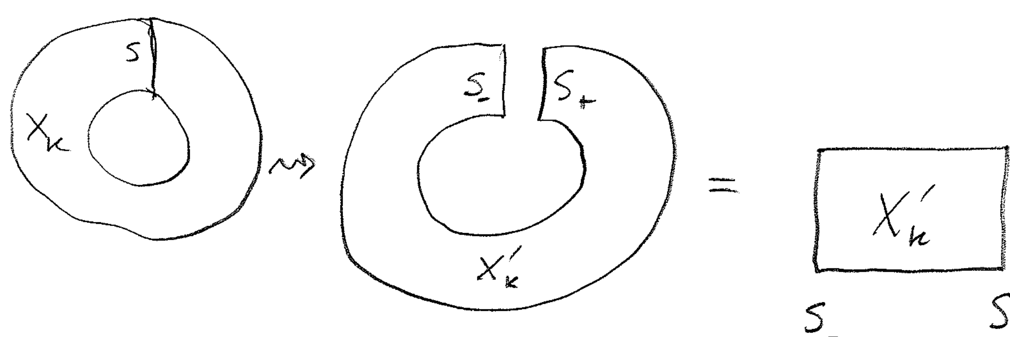
be a surjective homomorphism (unique up to $\underset{x \mapsto \pm x}{\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}}$)

Let $p : \widehat{X}_K \to X_K$ be the cover of X_K corresponding to $\ker(\varphi) \triangleleft \pi_1(X_K)$.

Can construct $\widetilde{X}_K$ explicitly:

cut X_K open along $S \Rightarrow X_K' \supset S_- \sqcup S_+$

schematic



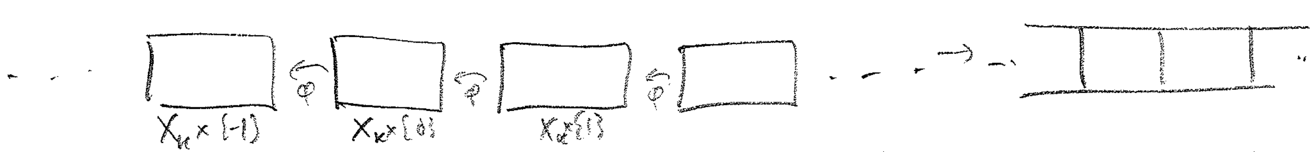
get X_K by gluing S_- to S_+ inside X_K'; say $\varphi : S_- \to S_+$ is gluing map

orientation on S gives $+$ & $-$ sides S_- & S_+.

Now take disjoint union

$$\coprod_{j \in \mathbb{Z}} X_K' \times \{j\} = X_K' \times \mathbb{Z}$$

Construct $\widehat{X}_K = X_K' \times \mathbb{Z} / \sim$ where $(x,j) \sim (\varphi(x), j-1)$ $\forall x \in S_-$



$\mathbb{Z} = \pi_1(X_K)/\ker(\varphi) \cong G(\widehat{X}_K, p)$ acts on $\widehat{X}_K$ here

simply by $n \cdot (y, j) = (y, n+j)$ quotient is X_K.

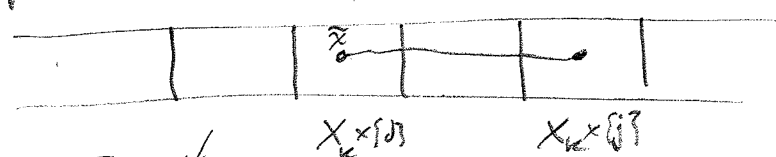
Let t be the generator $G(\widehat{X}_K, p) = \langle t \rangle$

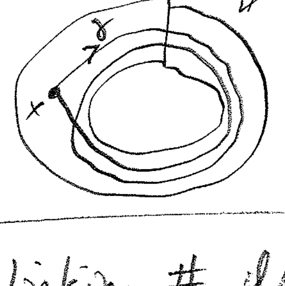
<u>Facts</u>

- $\langle t \rangle = G(\widehat{X}_K, p)$ acts on $\pi_1(\widehat{X}_K)^{ab}$
- $\pi_1(\widehat{X}_K)^{ab}$ can be viewed as a <u>module</u> over

  $\mathbb{Z}[\langle t \rangle] = \mathbb{Z}[t, t^{-1}]$, which we denote M_K.
- M_K is a finitely generated module over $\mathbb{Z}[t, t^{-1}]$ called the <u>Alexander invariant of K</u>.
- The ideal in $\mathbb{Z}[t, t^{-1}]$ that acts trivially on $\pi_1(\widehat{X}_K)^{ab}$ is principal gen. by $\Delta_K(t)$.

(*) <u>Observe</u>: $\varphi : \pi_1(X_K) \longrightarrow \mathbb{Z}$ can be described by intersections of the loops $\gamma \in \pi_1(X_K)$ w/ Σ, counted with sign:





then $\varphi(\gamma) = j$

$$\boxed{\Rightarrow \text{linking } \# \ lk(J, K) = \varphi_K(J)}$$