Math 428 4-28-11

Van Kampen's Theorem

Defn Given groups $\{G_\alpha\}_{\alpha\in J}$, the free product $\ast_{\alpha\in J} G_\alpha$ is the group whose elts are (possibly empty) reduced words: $g_1 \cdots g_k$ w/ $g_i \in G_{\alpha_i} \setminus \{1_{\alpha_i}\}$, $\alpha_i \neq \alpha_{i+1}$ $\forall i$.

- product: $(g_1 \cdots g_k)(g_1' \cdots g_\ell') = (g_1 \cdots g_k g_1' \cdots g_\ell')$ & reduce (multiply $(g_k g_1')$ if $g_k, g_1' \in G_\alpha$ some α & cancel if $g_k g_1' = 1_\alpha$ — continue)
- assoc. (see Hatcher)
- identity $= \emptyset$ $(= 1)$
- $(g_1 \cdots g_k)\cdot(g_k^{-1} \cdots g_1^{-1}) = 1$.

This has a universality property: Given any collection of homomorphisms $\{\varphi_\alpha : G_\alpha \to G\}$, $\exists!$ $\varphi : \ast_{\alpha\in J} G_\alpha \to G$ w/ $i_\alpha : G_\alpha \to \ast_{\alpha\in J} G_\alpha$ inclusion satisfying

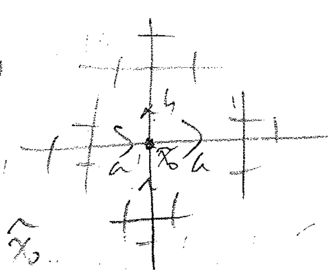
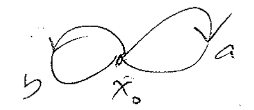
$$\varphi\, i_\alpha = \varphi_\alpha.$$

Ex $\mathbb{Z} \ast \mathbb{Z}$: Let $\mathbb{Z} \cong \langle a \rangle$, $\mathbb{Z} \cong \langle b \rangle$, we write

$F(a,b) = \langle a \rangle \ast \langle b \rangle \cong \mathbb{Z} \ast \mathbb{Z}$.

elts of $F(a,b)$ are words $a^{x_1} b^{y_1} \cdots a^{x_n} b^{y_n}$ (w/ x_1 & y_n possibly $\emptyset$) and empty word.

Note: $\pi_1(\infty) \cong \mathbb{Z} \ast \mathbb{Z}$. To see this, orient & label edges 

the universal covering is  

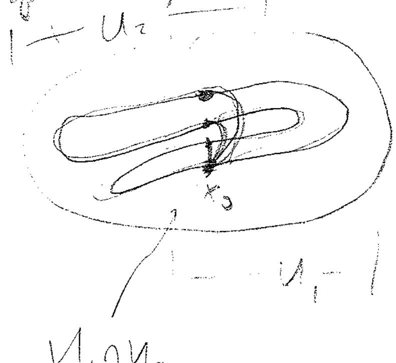
Any loop γ based at x_0 lifts to a path $\tilde\gamma$ starting at $\tilde x_0$ that is homotopic rel endpts to an embedded edge path — edges read off reduced word in a & b determined by $[\gamma]$ $\Rightarrow \pi_1(\infty) \to F(a,b)$ clearly an $\cong$.

This is a special case of Van Kampen's Theorem (though the proof is very different).

<u>Theorem</u> (Van Kampen) Suppose X is a union of path connected open sets $X = \bigcup_{\alpha \in J} U_\alpha$, $x_0 \in \bigcap_{\alpha \in J} U_\alpha$, & $\forall \alpha, \beta$ $U_\alpha \cap U_\beta$ p.c. Then $\Phi : \ast_{\alpha \in J} \pi_1(U_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$ is surjective. If $U_\alpha \cap U_\beta \cap U_\gamma$ is p.c. $\forall \alpha \beta \gamma$, then $\ker \Phi$ is the normal closure of $\{ i_{\alpha\beta}(\gamma) i_{\beta\alpha}(\gamma)^{-1} \mid \alpha, \beta \in J,\ \gamma \in \pi_1(U_\alpha \cap U_\beta) \}$.

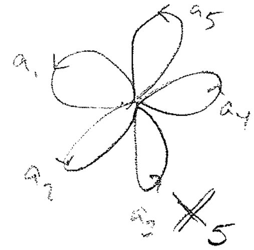
Here, $i_{\alpha\beta} : \pi_1(U_\alpha \cap U_\beta, x_0) \longrightarrow \pi_1(U_\alpha, x_0)$ is the hom. induced by inclusion and <u>normal closure</u> is smallest normal subgroup containing the elts.

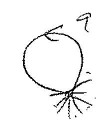
<u>Idea of proof</u>: For surjectivity, take any loop, partition the domain so each subinterval goes into one U_α. Then when switching open sets, connected back to x_0 w/ path:



For 2nd part, break homotopy up into little squares so that each maps into a U_α. Now view entire homotopy a finite sequence of small homotopies using squares, one at a time. Interface of 2 squares gives a "replacement" of $i_{\alpha\beta}(\gamma)$ w/ $i_{\beta\alpha}(\gamma)^{-1}$ ... <u>See Hatcher</u>. □

Ex: $X = X_k$ = wedge of k circles



set $U_i =$  $\underset{\text{open}}{\subset} X$

$$\pi_1(X_k) \cong \ast_{i=1}^{k} \pi_1(U_i)$$

$$= \ast_{i=1}^{k} \langle a_i \rangle =: F(a_1, \dots, a_k) \quad \text{free group on } a_1, \dots, a_k$$

$$\cong \ast_{i=1}^{k} \mathbb{Z} =: F_k$$

Note • $U_i \cap U_j = U_i \cap U_j \cap U_\ell$ = $\ast$ contractible

• U_i hom. eq. to S^1 

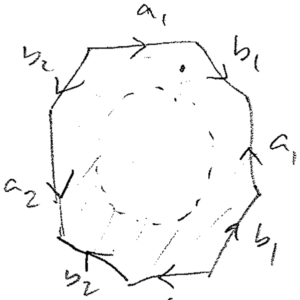
Ex: Already saw that if $k \neq 0$, $S_{g,k}$ h.e. to X_{2g+k-1} (similarly $M_{n,k}$, non-orient. surface, h.e. to X_{n+k-1}) so

$$\pi_1(S_{g,k}) \cong F_{2g+k-1}, \quad \pi_1(M_{n,k}) \cong F_{n+k-1}$$

What about S_g, M_n?

$\underline{\underline{S_g}}$

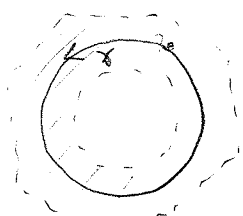
4g-gon sides identified

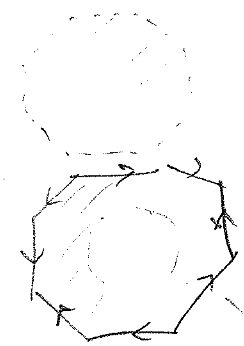


U_2 = interior of $4g$-gon

U_1 = $4g$-gon minus smaller disk

$U_1 \cap U_2$ h.e. to S^1.





h.e. to  wedge of 2g circles

$$\text{V.K.} \Rightarrow \pi_1(S_g) \cong \pi_1(U_1) * \underset{\text{trivial}}{\pi_1(U_2)} \Big/ \langle\langle \{ i_{12}(\gamma^k)\, i_{21}(\gamma^{-k}) \}_{k \in \mathbb{Z}} \rangle\rangle$$

$\langle\langle - \rangle\rangle$ = normal closure.

$$= \pi_1(U_1) \Big/ \langle\langle \gamma^k \rangle\rangle = \pi_1(U_1) \Big/ \langle\langle \gamma \rangle\rangle$$

$$= F(a_1, b_1, \dots, a_g, b_g) \Big/ \langle\langle a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle\rangle$$

$$= F(a_1, b_1, \dots, a_g, b_g) \Big/ \langle\langle [a_1, b_1] \cdots [a_g, b_g] \rangle\rangle$$

$[\,,\,]$ = commutator

"Shorthand" is described by a presentation:

Given a group G, a presentation for G is an expression

$\langle A \mid R \rangle$, write $G = \langle A \mid R \rangle$

where A is a set, R some subset of $F(A) = \ast_{a\in A} \langle a \rangle \cong \ast_{a \in A} \mathbb{Z}$ $(\langle a \rangle = \mathbb{Z})$

so that $G \cong F(A)/\langle\langle R \rangle\rangle$, $\langle\langle R \rangle\rangle$ = normal closure of R.

<u>Ex</u>

- $\langle A \mid - \rangle = F(A)$
- $\langle a,b \mid [a,b] \rangle = \mathbb{Z}\oplus\mathbb{Z}$ to see this, construct a hom.

(by univ. props) $F(a,b) \xrightarrow{\varphi} \mathbb{Z}\oplus\mathbb{Z}$ $[a,b] \in \ker(\varphi)$ so get an induced hom.

$a \longmapsto (1,0)$

$b \longmapsto (0,1)$

$$F(a,b)/\langle\langle [a,b] \rangle\rangle \underset{\bar\varphi^{-1}}{\overset{\bar\varphi}{\rightleftarrows}} \mathbb{Z}\oplus\mathbb{Z}$$

$F(a,b)/\langle\langle [a,b] \rangle\rangle$ is an abelian group — the generators commute, so get an inverse hom $\bar\varphi^{-1}(1,0) = a$, $\bar\varphi^{-1}(0,1) = b$,

$\bar\varphi$ is an $\cong$. $\Longleftarrow$

- $\langle a_1, \ldots, a_k \mid \{[a_i, a_j]\}_{i,j=1}^{k} \rangle \cong \mathbb{Z}^k$.
- $\pi_1(S_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle$

<u>Exercise</u> compute a presentation for $\pi_1(M_n)$, $n > 0$. using V.K.

<u>Abelianization</u>: Given a group G, the quotient by $[G,G] = \langle \{[g,h]\}_{g,h\in G} \rangle \trianglelefteq G$ is called the <u>abelianization</u> $G^{ab} = G/[G,G]$.

If $G = \langle A | R \rangle$, then $G^{ab} = \langle A \,|\, R, \{[a,a']\}_{a,a'\in A} \rangle$

<u>Ex</u> $(F_k)^{ab} \cong \mathbb{Z}^k$

G^{ab} is the "largest" abelian quotient of G:

If $\varphi : G \longrightarrow H$ is a homomorphism w/ H abelian, then $\varphi([G,G]) = 1$ so $\exists\ \bar{\varphi} : G^{ab} \longrightarrow H$ w/

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr] & & H \\ & G^{ab} \arrow[ur, "\bar{\varphi}"'] \arrow[u, phantom, "\circ"] & \end{tikzcd}$$

$(X,x) \rightsquigarrow \pi_1(X,x) \rightsquigarrow (\pi_1(X,x))^{ab}$ is an <u>invariant</u> of (X,x).

$(G \cong K \Rightarrow G^{ab} \cong K^{ab})$

<u>Exercise</u> show $(\pi_1(S_g))^{ab} = \mathbb{Z}^{2g}$, $(\pi_1(M_n))^{ab} = \mathbb{Z}^{n-1} \oplus \mathbb{Z}/2\mathbb{Z}$

<u>so</u> for compact connected surfaces M (w/o boundary) $\pi_1(M)$ (and even $\pi_1(M)^{ab}$) distinguishes the homeomorphism type.

<u>Presentation matrices</u>: Another view of G^{ab}: replace $A = \{a_1, \dots, a_k\}$ by <u>commuting</u> generators, call them $\{A_1, \dots, A_k\}$. If $R = \{r_1, \dots, r_l\}$, this is replaced by $R_1, \dots, R_l$ which each have the form

$$R_i = A_1^{x_{i1}} A_2^{x_{i2}} \cdots A_k^{x_{ik}} \text{ for } i = 1, \dots, l, \text{ for some } (x_{i1} \cdots x_{ik}) \in \mathbb{Z}^k.$$

Then $G^{ab} \cong \mathbb{Z}^k \Big/ \langle \{(x_{i1} \cdots x_{ik})\}_{i=1}^{l} \rangle$ (subgroup gen. is automatically normal)

Set $M = (x_{ij}) = \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{l1} & \cdots & x_{lk} \end{pmatrix}$

right mult. by M defines a homomorphism

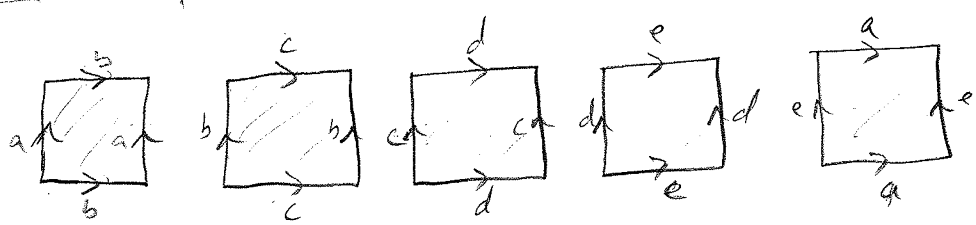
$$M : \mathbb{Z}^l \longrightarrow \mathbb{Z}^k$$

integer row & column operations on M, $M \rightsquigarrow M'$, then

$$\mathbb{Z}^k/(\mathbb{Z}^l)M \cong \mathbb{Z}^k/(\mathbb{Z}^l)M'.$$

FACT (Fund. Thm of f.g. ab. grps) can always diagonalize M w/ $\mathbb{Z}$-row & column operations.

Exercise compute presentation for $\pi_1(X)$, where:

$X =$ 

Is $\pi_1(X)$ abelian? (Hint:

Homomorphism from $G = \langle A | R \rangle$. It is easy to decide whether or not a map $\varphi : A \longrightarrow H$, H a group, extends to a homomorphism of G:

$\varphi \Rightarrow \hat{\varphi} : F(A) \longrightarrow H$ a unique extension to a homomorphism.

$\hat{\varphi}$ descends to $\varphi : \langle A | R \rangle = F(A)/\langle\langle R \rangle\rangle \longrightarrow H \iff \hat{\varphi}(R) = 1$

Exercise Prove that $\pi_1(S_g)$, $\pi_1(M_n)$ are non abelian for $g \geq 2$ & $n \geq 3$.

Hint: construct a hom. onto a nonabelian gp.