Math 428 4/21/11

Defn A <u>pointed covering map</u> is a map $p:(\tilde X,\tilde x_0)\to(X,x_0)$ s.t. $p:\tilde X\to X$ is a covering map and $p(\tilde x_0)=x_0$. The homeomorphism h of the proposition is called an <u>isomorphism</u> of (pointed) covering spaces, and we say that $p_1:\tilde X_1\to X$ and $p_2:\tilde X_2\to X$ are isomorphic covering spaces (and $p_1:(\tilde X_1,\tilde x_1)\to(X,x)$, $p_2:(\tilde X_2,\tilde x_2)\to(X,x)$ are isomorphic pointed covering spaces).

If $p:\tilde X\to X$ is a covering space, an isomorphism $h:\tilde X\to\tilde X$ is called a <u>covering transformation</u>, that is $h:\tilde X\to\tilde X$ is a homeomorphism with $p=h\circ p$. The covering transformations of $p:\tilde X\to X$ form a group $G(\tilde X,p)$ called the <u>covering group</u> (or <u>deck group</u>) of $p:\tilde X\to X$.

The covering $p:\tilde X\to X$ is called <u>regular</u> if $G(\tilde X,p)$ acts <u>transitively</u> on the fibers of p.

[that is, $\forall\, x\in X$, $\tilde x_1,\tilde x_2\in p^{-1}(x)$ $\exists\, h\in G(\tilde X,p)$ st. $h(\tilde x_1)=\tilde x_2$.]

Theorem III.64. Let X be p.c.l.p.c, $x\in X$, $p:\tilde X\to X$ a connected & simply connected covering and $\tilde x\in p^{-1}(x)$. Then $\exists$ a bijection between isomorphism classes of pointed connected coverings and subgps of $\pi_1(X,x)$ determined by

$$\{p_0:(\tilde X_0,\tilde x_0)\to(X,x)\}/_{\text{iso.}} \longleftrightarrow \{H<G\}$$

$$[p_0:(\tilde X_0,\tilde x_0)\to(X,x)] \longleftrightarrow (p_0)_*(\pi_1(\tilde X_0,\tilde x_0)).$$

Furthermore,

(1) conjugate subgps correspond to different choices of base points:

$$(H,\gamma H\gamma^{-1}) \longleftrightarrow (p_0:(\tilde X_0,\tilde x_0)\to(X,x),\ p_0:(\tilde X_0,\tilde x_1)\to(X,x))$$

when γ lifts to $\tilde\gamma$ in $\tilde X_0$ w/ $\tilde\gamma(0)=\tilde x_1$, $\tilde\gamma(1)=\tilde x_0$.

(2) $p_0:\tilde X_0\to X$ is a regular covering iff $(p_0)_*(\pi_1(\tilde X_0,\tilde x_0))\trianglelefteq\pi_1(X,x)$.

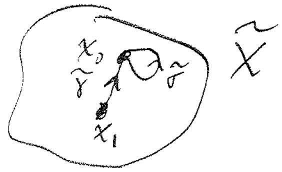
(3) $G(\tilde X_0,p_0)\cong N_{\pi_1(X,x)}((p_0)_*(\pi_1(\tilde X_0,\tilde x_0)))/(p_0)_*(\pi_1(\tilde X_0,\tilde x_0))$.

$$h\longmapsto[p_0\circ\gamma_h]\quad\text{where } \gamma_h \text{ is a path from } \tilde x_0 \text{ to } h(\tilde x_0).$$

(4) In particular, $G(\tilde X,p)\cong\pi_1(X,x)$, with $X\cong\tilde X/G(\tilde X,p)$.

<u>Proof</u> The correspondence & bijectivity is precisely III.62, 63.

For (1), just observe that conjugation is realized by changing basepoints



(2) follows from (1): If normal subgp, then For any two base points $\hat{x}_1, \hat{x}_2 \in p_0^{-1}(x)$,
$(p_0)_*(\pi_1(\hat{X}_0, \hat{x}_1)) = (p_0)_*(\pi_1(\hat{X}_0, \hat{x}_2))$ So, by III.63, $\exists\ h : (\hat{X}_0, \hat{x}_1) \to (\hat{X}_0, \hat{x}_2)$
isomorphism of covers $\Rightarrow h \in G(\hat{X}_0, \bar{p}_0)$ & the group acts transitively.
Conversely, $G(\hat{X}_0, \bar{p}_0)$ acts transitively on $p_0^{-1}(x)$ so by correspondence & (1)
all conjugate subgroups are the same $\Rightarrow (p_0)_*(\pi_1(\hat{X}_0, \hat{x}_0)) \lhd \pi_1(X, x)$.

(3) same idea as in (2): any elt of normalizer gives a covering transformation.

(4) Follows from (3) since $p_*(\pi_1(\tilde{X}, \tilde{x})) = \{1\} \lhd \pi_1(X, x)$. and
$G(\tilde{X}, p)$ transitive on fibers of p makes $\bar{p}$ bijection, Check its a homeomorphism □.

$$\begin{tikzcd}
\tilde{X} \arrow[d, "p"'] \arrow[dr] & \\
X & \tilde{X}/G(\tilde{X}, p) \arrow[l, "\bar{p}"]
\end{tikzcd}$$

Get Galois correspondence picture

$$\begin{tikzcd}
\tilde{X} \arrow[d] & & \{1\} \arrow[d, dash] \\
\tilde{X}_H \arrow[d, dotted] \arrow[rr, leftrightarrow] & & H \arrow[d, dash] \\
X & & \pi_1(X, x)
\end{tikzcd}$$

subgrp containment corresponds to covering

Given a group G and a top. space X, an action of G on X is called a <u>covering action</u> if $\forall\ x \in X\ \exists$ nbhd U_x of x s.t. $g \cdot U_x \cap U_x = \emptyset$ if $g \neq 1$. Action of $G(\tilde{X}, p)$ on $\tilde{X}$, w/ $p: \tilde{X} \to X$ a covering space is a covering action when $\tilde{X}$ is p.c.

<u>Proposition III.45</u>: If $G \circlearrowright X$ is a covering action then the quotient map to the orbit space

$$p: X \longrightarrow X/G$$
$$x \longmapsto G \cdot x$$

is a covering map and $G = G(X, p)$.

<u>proof</u>: Observe that $\forall\ x \in X$.

$$p^{-1}(p(U_x)) = \coprod_{g \in G} g \cdot U_x \subseteq_{\text{open}} X.$$

so $p(U_x)$ is open. By assumption $p|_{gU_x}: gU_x \to p(U_x)$ is a bijection and easily seen to be a homeomorphism. So, $p(U_x)$ is an evenly covered nbhd of $p(x)$. $\square$.

<u>Ex</u> Consider $S^3 \subset \mathbb{C}^2$ ~~as~~ $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$.

For any $p, q \in \mathbb{Z}_+$, $\gcd(p, q) = 1$, consider the action of $\mathbb{Z}/p\mathbb{Z}$ on S^3, generated by

$$(z_1, z_2) \longmapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$$

. More explicitly

$$T_{p,q} = \begin{pmatrix} e^{2\pi i/p} & 0 \\ 0 & e^{2\pi i q/p} \end{pmatrix} \in GL_2\mathbb{C} \circlearrowright \mathbb{C}^2,$$ $T_{p,q}$ preserves S^3,

$$\langle T_{p,q} \rangle \cong \mathbb{Z}/p\mathbb{Z}.$$

Note $\gcd(p,q)=1 \Rightarrow \langle e^{\frac{2\pi i}{p}} \rangle = \langle e^{\frac{2\pi i q}{p}} \rangle < S^1$

$T^r_{p,q}(z_1, z_2) = (z_1, z_2)$

$\iff z_1 = e^{\frac{2\pi i r}{p}} z_1 \ , \ z_2 = e^{\frac{2\pi i r q}{p}} z_2$

$\iff e^{\frac{2\pi i r}{p}} = 1$ or $e^{\frac{2\pi i r q}{p}} = 1$

z_1, z_2 not both 0.

$\iff e^{\frac{2\pi i r}{p}} = e^{\frac{2\pi i r q}{p}} = 1 \iff T^r_{p,q} = I$

so no nontrivial elt of $\langle T_{p,q} \rangle$ fixes a point.

Since this group is Hausdorff and $\langle T_{p,q} \rangle$ is finite follows that this is a covering group action.

$S^3 / \langle T_{p,q} \rangle =: L(p,q)$ Lens space.

$f_{p,q} : S^3 \longrightarrow L(p,q)$ quotient map is a covering map,

$G(S^3, f_{p,q}) \cong \mathbb{Z}/p\mathbb{Z}$ & since $\pi_1(S^3) = \{1\}$ we have:

$\pi_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$.

EX $\mathbb{Z}/2\mathbb{Z} \curvearrowright S^n$ with generator sending $x \mapsto -x$

$p : S^n \longrightarrow S^n/\mathbb{Z}/2\mathbb{Z} =: \mathbb{RP}^n$ (= lines through 0 in $\mathbb{R}^{n+1}$).

$G(S^n, p) \cong \mathbb{Z}/2\mathbb{Z}$ since $\pi_1(S^n) = \{1\}$ for $n \geq 2$, we have

$\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$.

Exercise

$$H_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\} < SL_3\mathbb{Z}$$

$\wedge$

$$H = \{ \quad \cdots \quad \mathbb{R} \} < SL_3\mathbb{R}.$$

$\cong$

$\mathbb{R}^3$

$H_{\mathbb{Z}}$ acts on $H \cong \mathbb{R}^3$. prove that this is a covering group action. thus we have $\pi_1(H/H_{\mathbb{Z}}) \cong H_{\mathbb{Z}}$.

Exercise consider the group of homeomorphisms of $\mathbb{R}^2$ generated by $f(x,y) = (x+\frac{1}{2}, -y)$, $g(x,y) = (x, y+1)$. let $\langle f, g\rangle \subset \text{Homeo}(\mathbb{R}^2)$

(a) construct a homomorphism $D : \langle f, g \rangle \longrightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} < GL_2\mathbb{Z}$
$\cong \mathbb{Z}/2\mathbb{Z}$.

(b) show that $\ker(D) \cong \mathbb{Z}^2$, and that the induced action on $\mathbb{R}^2$ is the usual action of $\mathbb{Z}^2$ by translations.

(c) show that $\langle f, g \rangle \curvearrowright \mathbb{R}^2$ is a covering action, so

$$\pi_1(\mathbb{R}^2/\langle f, g\rangle) \cong \langle f, g \rangle$$

(d) $\mathbb{R}^2/\langle f, g\rangle$ is a compact surface without boundary, which one is it?