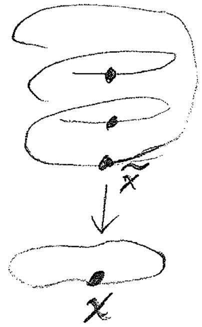
Proposition III.59 If $p:\tilde{X}\to X$ is a covering map and $X,\tilde{X}$ are path connected, then $\forall\, x,x'\in X$, $|p^{-1}(x)|=|p^{-1}(x')|=[\pi_1(X,x):p_*(\pi_1(\tilde{X},\tilde{x}))]$ for any $\tilde{x}\in p^{-1}(x)$.

proof

Let



$[\gamma],[\sigma]\in\pi_1(X,x)$, let $\tilde{\gamma},\tilde{\sigma}$ be lifts of γ,σ st. $\tilde{\gamma}(0)=\tilde{\sigma}(0)=\tilde{x}$.
Note $\tilde{\gamma}(1)\in p^{-1}(x)$ and depends only on $[\gamma]$ by Corollary III.57.

Observe that $\tilde{\gamma}(1)=\tilde{\sigma}(1) \iff \tilde{\gamma}\bar{\tilde{\sigma}}$ is a loop (based at $\tilde{x}$).

$$\iff [\gamma][\bar{\sigma}]=[\gamma][\sigma]^{-1}\in p_*(\pi_1(\tilde{X},\tilde{x}))$$

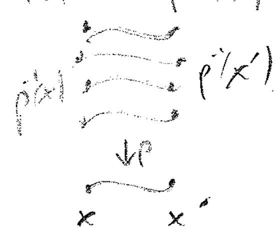
$$\iff p_*(\pi_1(\tilde{X},\tilde{x}))[\gamma]=p_*(\pi_1(\tilde{X},\tilde{x}))[\sigma].$$

this sets up a bijection between

$p^{-1}(x) \longleftrightarrow$ cosets of $p_*(\pi_1(\tilde{X},\tilde{x}))$ in $\pi_1(X,x)$

so $|p^{-1}(x)|=[\pi_1(X,x):p_*(\pi_1(\tilde{X},\tilde{x}))]$

Finally, if σ is a path from x to x', the lifts of $\tilde{\sigma}$ define a bijection between $p^{-1}(x)$ & $p^{-1}(x')$



$\square$

$|p^{-1}(x)|$ is called the number of <u>sheets</u> of the covering (possibly ∞)

We would like to know when a map $f:Y\to X$ lifts to a covering space $p:\tilde{X}\to X$. One answer is provided in a fairly general setting.

<u>Defn</u> Y is <u>locally path connected</u> if for every pt $y \in Y$ and every nbhd U of y, $\exists$ <u>path connected nbhd</u> $y \in V \subset U$.

> Ex
> - manifolds are locally path connected
> - $\mathbb{Q} \times \mathbb{R} \cup \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ is path connected, but only locally path connected at pts of $\mathbb{R} \times \{0\}$

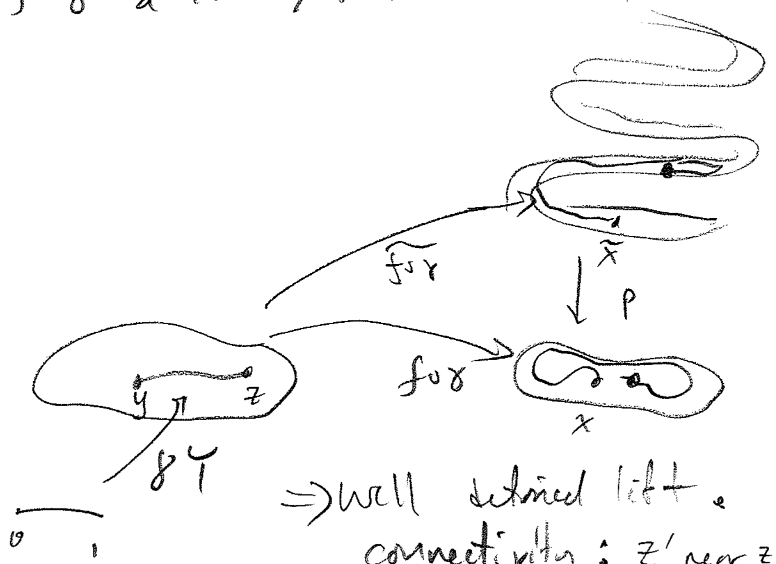
<u>Proposition III.60</u> Suppose Y is path connected & locally path connected, $p: \hat{X} \to X$ is a covering map and $y \in Y$, $x \in X$, $\tilde{x} \in p^{-1}(x)$. A map $f: (Y, y) \to (X, x)$ has a lift $\tilde{f}: (Y, y) \to (\hat{X}, \tilde{x})$ if and only if

$$f_*(\pi_1(Y, y)) \subseteq p_*(\pi_1(\hat{X}, \hat{x})).$$

<u>sketch of proof</u>: One direction is clear: if $\tilde{f}$ is a lift, then $p\tilde{f} = f$ so

$$\underbrace{p_* \tilde{f}_*(\pi_1(Y, y))}_{f_*(\pi_1(Y, y))} \subseteq p_*(\pi_1(\tilde{X}, \hat{x}))$$

For the other direction, we just construct the lift by lifting paths: $\forall$ $z \in Y$, let γ be a path from z to y and $\widetilde{f \circ \gamma}$ a lift of $f \circ \gamma$ to $\hat{X}$ w/ $\widetilde{f \circ \gamma}(0) = \tilde{x}$, set $\hat{f}(z) = \widetilde{f \circ \gamma}(1)$.



Any other path σ connects y to z has $\gamma\bar{\sigma}$ a loop so $f \circ (\gamma\bar{\sigma})$ is a loop at $\widetilde{f \circ \gamma\bar{\sigma}}$ is a loop. hence $\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \sigma}(1)$.

$\Rightarrow$ well defined lift. Continuous by local path connectivity: z' near z is connected to z by a short path, image by f lies in $U_{f(z)}$ an evenly covered open set & can lift by inverse homeo. $\square$

Now we know when there is a lift, the next prop. tells us when it is unique.

Proposition III.61 Suppose $p: \tilde{X} \to X$ is a covering map, $f: Y \to X$ a map, $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ two lifts. If Y is connected & $\tilde{f}_1(y) = \tilde{f}_2(y)$ for some $y \in Y$, then $\tilde{f}_1 = \tilde{f}_2$.

proof Similar to Lemma III.46 – set where $\tilde{f}_1 = \tilde{f}_2$ is open & closed $\square$

Classification of covers & Galois correspondence

Suppose from now on that X is path connected & locally path connected (p.c. l.p.c.). We consider connected (equiv. path connected) covers $p: \tilde{X} \to X$. [$\tilde{X}$ is automatically l.p.c. & as such, p.c. & conn. are the same]

Suppose X admits a simply connected covering space

$$p: \tilde{X} \to X$$

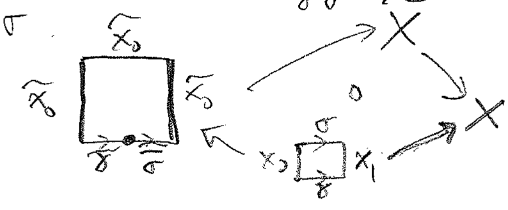
Pick a base point $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$, then we have a bijection

$$\{\gamma : [0,1] \to X \mid \gamma(0) = x_0\} / \simeq_\partial \longrightarrow \tilde{X}$$

$$\gamma \longmapsto \tilde{\gamma}(1) \quad \text{, where } \tilde{\gamma} \text{ is a lift of } \gamma \text{ w/ } \tilde{\gamma}(0) = \tilde{x}_0.$$

[In fact, under suitable local conditions on X, e.g. if X is mfld or graph, such $\tilde{X}$ can be constructed by appropriately topologizing set on left]

- onto since $\tilde{X}$ path connected
- 1-1 since $\tilde{\gamma}(1) = \tilde{\sigma}(1) \Rightarrow \gamma\bar{\sigma}$ is a loop w/ $\widetilde{\gamma\bar{\sigma}} = \tilde{\gamma}\bar{\tilde{\sigma}}$ a lift. $\tilde{X}$ simply connected implies $\widetilde{\gamma\bar{\sigma}} \simeq_\partial \text{const.} \Rightarrow \gamma \simeq_\partial \sigma$.



$x_1 = \gamma(1) = \sigma(1)$.

Proposition III.62 Let $p:\tilde{X}\to X$ be as above. Then for every $H<\pi_1(X,x_0)$, $\exists$ p.c. covering space $p_H:\tilde{X}_H\to X$, $\tilde{x}_0^H\in p_H^{-1}(x_0)$ s.t. $(p_H)_*(\pi_1(\tilde{X}_H,\tilde{x}_0^H))=H$. Furthermore

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[rr, bend right, "p"'] & \tilde{X}_H \arrow[r, "p_H"] & X \end{tikzcd}$$

proof: construct $\tilde{X}_H$ as quotient of $\tilde{X}=\{\gamma:[0,1]\to X\}/\simeq_\partial$ by relation $\gamma\sim_H\sigma$ if $[\gamma\bar{\sigma}]\in H<\pi_1(X,x_0)$. □

Ex: $p:\mathbb{R}\longrightarrow S^1$, $p(t)=e^{2\pi i t}$. $\pi_1(S^1,1)=\langle[\gamma_1]\rangle$ w/ $\gamma_1(t)=e^{2\pi i t}$, $t\in[0,1]$

$H<\pi_1(S^1,1)$ is cyclic, so $H=\langle[\gamma_1]^k\rangle$, some $k\in\mathbb{Z}_+$.

In identification $\mathbb{R}\longleftrightarrow\{\gamma:[0,1]\to S^1\}/\simeq$,

$r \longleftrightarrow [\gamma_r]$, where $\gamma_r(t)=e^{2\pi i r t}$

then $\tilde{X}_H$ is defined by $\gamma_r\sim_H\gamma_s$ if $[\gamma_r\bar{\gamma}_s]\in\langle[\gamma_1]^k\rangle=\langle[\gamma_k]\rangle=H$

$\gamma_r\bar{\gamma}_s\simeq_\partial\gamma_{r-s}$ so $[\gamma_r\bar{\gamma}_s]\in H \iff r-s\in k\mathbb{Z}$

so $\tilde{X}_H=\mathbb{R}/k\mathbb{Z}\longrightarrow\mathbb{R}/\mathbb{Z}=S^1$.

Moreover, $\tilde{X}_H$ is essentially unique.

Proposition III.63 Let X be as above, $p_1:\tilde{X}_1\to X$, $p_2:\tilde{X}_2\to X$ connected covering spaces. $(p_1)_*(\tilde{X}_1,\tilde{x}_1)=(p_2)_*(\tilde{X}_2,\tilde{x}_2)<\pi_1(X,*)$ iff $\exists$ homeomorphism $h:(\tilde{X}_1,\tilde{x}_1)\longrightarrow(\tilde{X}_2,\tilde{x}_2)$ s.t. $p_1=h\circ p_2$.

proof:

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, dashed, bend right, "\tilde{p}_1"'] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[ll, dashed, bend right, "\tilde{p}_2"'] \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

lifts $\tilde{p}_1,\tilde{p}_2$ exist by III.60 w/ $\tilde{p}_1(\tilde{x}_1)=\tilde{x}_2$, $\tilde{p}_2(\tilde{x}_2)=\tilde{x}_1$

$\tilde{p}_2\tilde{p}_1$ is a lift of p_1, so is the identity so $\tilde{p}_2\tilde{p}_1=Id_{\tilde{X}_1}$. similarly $\tilde{p}_1\tilde{p}_2=id_{\tilde{X}_2}$.

Converse is obvious □