This requires a little more story...

A *retraction* $r: X \to A$ is a continuous map of a top. space $X$ & a subspace $A \subset X$ s.t. $r(a) = a$ $\forall a \in A$.

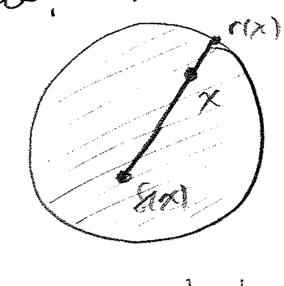
**Proposition III.49** If $x \in A \subset X$ and $r: X \to A$ is a retraction, then $r_*: \pi_1(X, x) \to \pi_1(A, x)$ is surjective & $i_*: \pi_1(A,x) \to \pi_1(X,x)$ is injective, where $i: A \to X$ is inclusion □

**proof** (exercise) □ (special case of def. retraction $r_* = (i_*)^{-1}$).

**Corollary III.50.** $\nexists$ a retraction $r: \bar{B}^2 \to S^1$

**proof**: $\pi_1(\bar{B}^2) = \{1\}$ & $\pi_1(S^1) \cong \mathbb{Z}$ □

**proof of BFPT.** Suppose $\exists f: \bar{B}^2 \to \bar{B}^2$ w/ no fixed point. define $r: \bar{B}^2 \to S^1$ as follows. $\forall x \in \bar{B}^2$, let $L_x$ be the line through $x$ & $f(x)$ WELL DEFINED since $f(x) \neq x$. oriented from $f(x)$ toward $x$. Define $r(x) \in S^1$ to be the 1st point of intersection of $L_x$ w/ $S^1$ s.t. $r(x) \geq x$ on $L_x$ — so $r(x) = x$ if $x \in S^1$. Check continuous. This is a retraction, contradicting III.49 □




Another application is:

**Fundamental Theorem of algebra III.51** Any nonconstant polynomial $f(z) \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.

Proof WLOG, write $p(z)=z^n+a_{n-1}z^{n-1}+\dots+a_1z+a_0$ w/ $n>0$.
Assume $p(z)$ has no roots in $\mathbb{C}$. $\forall\, r\geq 0$ define

$$\gamma_r(t)=\frac{p(re^{2\pi i t})/p(r)}{|p(re^{2\pi i t})/p(r)|}$$ which is a loop based at $1\in S^1\ \forall r$.

since $\gamma_0(t)$ is constant, $[\gamma_r]=[\gamma_0]=e\in\pi_1(S^1)\cong\mathbb{Z}$. $\forall r>0$
(b/c $\gamma_r\simeq_\partial\gamma_0\ \forall\, r\geq 0$)

Now consider the 1-parameter family of polynomials

$$P_s(z)=z^n+s(a_{n-1}z^{n-1}+\dots+a_1z+a_0),\quad s\in[0,1].$$

This give loops $\gamma_{r,s}$ defined via $p_s$ as $\gamma_r=\gamma_{r,1}$ is via $p=p_1$

If $r$ is sufficiently large (at least $|a_0|+\dots+|a_{n-1}|+1$)

Check: $P_s$ has no roots on $\{z \mid |z|=r\}$, for any $s$, then

$$\gamma_{r,0}(t)=\frac{r^ne^{2\pi i n t}/r^n}{|r^n/r^n|}=e^{2\pi i n t},\quad [\gamma_{r,0}]\neq e \text{ in } \pi_1(S^1,1)$$

and since $\gamma_{r,0}\simeq_\partial\gamma_r$, we have a contradiction $\square$

...And another...

<u>Borsuk-Ulam Theorem ($n=2$) II. 52</u> If $f:S^2\to\mathbb{R}^2$ is continuous, $\exists\, x\in S^2$ s.t. $f(x)=f(-x)$. (i.e. $f$ identifies a pair of antipodes)

Proof: sketch – assume not, set $g(x)=\dfrac{(f(x)-f(-x))}{|f(x)-f(-x)|}$, observe $g:S^2\to S^1$ and $g(-x)=-g(x)$, check that $g_*[\text{equator}]$ is non trivial in $\pi_1(S^1)$. This contradicts fact $[\text{equator}]=1$ in $\pi_1(S^2)$ $\square$

**Proposition III.54** $\pi_1(X\times Y,(x,y))\cong\pi_1(X,x)\times\pi_1(Y,y)$.

proof viewing $X\times\{y_0\},\{x_0\}\times Y\subset X\times Y$, the projections
$p_X: X\times Y\to X$ & $p_Y: X\times Y\to Y$, clearly
$(p_X)_*\times(p_Y)_*:\pi_1(X\times Y,(x,y))\longrightarrow\pi_1(X,x)\times\pi_1(Y,y)$ is onto.
$([\gamma],[\sigma])$ is the image of $[\gamma\times\sigma]$.
If $\gamma\simeq_\partial c_1$, $\sigma\simeq_\partial c_2$, then $\gamma\times\sigma\simeq_\partial c_1\times c_2$, so
ker $(p_X)_*\times(p_Y)_*$ is trivial $\square$

**Corollary III.55** $\pi_1(T^n)\cong\mathbb{Z}^n$ $\forall n\geq 1$

proof: $T^n\cong S^1\times S^1\times\ldots\times S^1$, induct on previous prop $\square$

Ex $\pi_1(S^n\times S^m)\cong\{1\}$ $\forall$ $m,n\geq 2$.

**Proposition III.53** $\mathbb{R}^2\cong\mathbb{R}^n\Rightarrow n=2$.

proof: We have already seen that $\mathbb{R}^n\setminus\{*\}$ is homotopy equivalent to $S^{n-1}$. So $\mathbb{R}^2\cong\mathbb{R}^n\Rightarrow S^1$ homotopy equiv to $S^{n-1}$.
If $n=1$, $S^{n-1}=S^0\cong\{\pm1\}$, disconnected $\downarrow\uparrow$
If $n>2$, $\pi_1(S^{n-1})=\{1\}$ $\downarrow\uparrow$

So, $n=2$. $\square$

Rk $S^k$ disconnected only for $k=0\Rightarrow[\mathbb{R}^1\cong\mathbb{R}^n\Rightarrow n=1]$

<u>Defn</u> A map $p: \tilde{X} \to X$ is called a <u>covering map</u> if $\forall\ x \in X$, $\exists$ nbhd $U_x$ of $x$ st

$$(*)\begin{cases} p^{-1}(U_x) = \coprod_{\alpha \in J} V_x^\alpha \quad w/\ V_x^\alpha \subset \tilde{X} \text{ open and} \\ p|_{V_x^\alpha}: V_x^\alpha \to U_x \text{ a homeomorphism } \forall \alpha \in J \end{cases}$$

Also say $p: \tilde{X} \to X$ is a covering space or just that $\tilde{X}$ is a covering space of $X$.

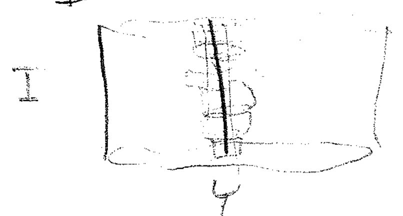
$U_x$ is called an <u>evenly covered</u> <u>nbhd</u> of $x$

Ex

The key property of covering spaces is <u>homotopy lifting</u>!

<u>Lemma III.56</u> If $p: \tilde{X} \to X$ is a covering map, $f: Y \to X$ is a map with lift $\tilde{f}: Y \to \tilde{X}$, and $H: Y \times I \to X$ is a homotopy of $f$ (rel $A$), then $\exists !$ lift $\tilde{H}: Y \times I \to X$ of $H$ which is a homotopy of $\tilde{f}$ (rel $A$).

<u>proof</u> Same idea as in proof of Lemma III.516: Exact same proof gives $\exists$ a unique lift of $\tilde{H}|_{\{y\}\times I}$ $\forall\ y \in Y$. In fact, compactness of $I \Rightarrow \exists$ nbhd $W$ of $y$ in $Y$ so we can construct a lift $\tilde{H}|_{W \times I}$.



So, we have $\tilde{H}$ and its continuous on open sets $W \times I$ that cover all of $Y \times I$, so $\tilde{H}$ continuous. (unique b/c unique on $\tilde{H}|_{\{y\}\times I}$). $\square$

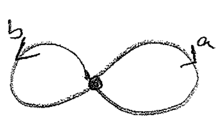
<u>Corollary III.57</u> If $p: \tilde{X} \to X$ is a covering map, $\gamma: [0,1] \to X$ a path, $\tilde{x} \in p^{-1}(\gamma(0))$, then $\exists !$ lift $\tilde{\gamma}: [0,1] \to \tilde{X}$. If $H$ is a homotopy of $\gamma$ (rel endpts) then $\exists !$ lift $\tilde{H}$ which is a homotopy of $\tilde{\gamma}$ (rel end pts). $\square$

<u>Proposition III.58</u> If $p: \tilde{X} \to X$ is a covering space, $x \in X$, $\tilde{x} \in p^{-1}(x)$ then $p_*: \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$ is <u>injective</u>. $p_*(\pi_1(\tilde{X}, \tilde{x})) = \{[\gamma] \in \pi_1(X, x) \mid \gamma$ lifts to a <u>LOOP</u> in $\tilde{X}$ based at $\tilde{x}\}$.

<u>proof</u> exercise from III.57. — compare with calculation of $\pi_1(S^1)$ $\square$.

**Ex** ① $p : \mathbb{R} \longrightarrow S^1$ is a covering map. Other covering maps of $S^1$ are $p_n : S^1 \longrightarrow S^1$ given by $p_n(z) = z^n$ (viewing $S^1 \subset \mathbb{C}$) $n > 0$

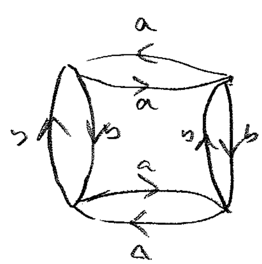
$(p_n)_* : \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$ is given by $(p_n)_*([\gamma_j]) = [\gamma_{jn}]$ when $\gamma_j(t) = e^{2\pi i j t}$, $t \in [0,1]$. That is, w.r.t. $\pi_1(S^1, 1) \cong \mathbb{Z}$, we have $(p_n)_*(j) = nj \;\; \forall\, j \in \mathbb{Z}$.

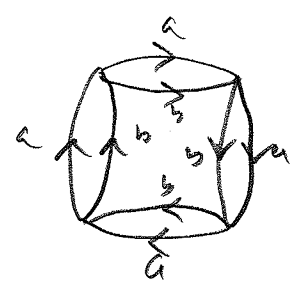
② $X =$ 

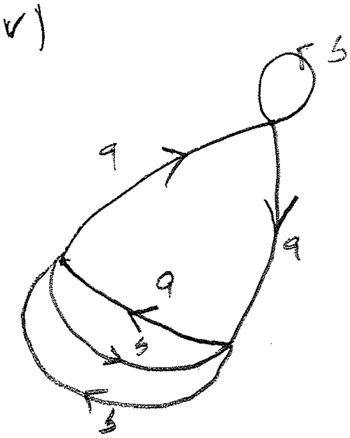
wedge of 2 circles.

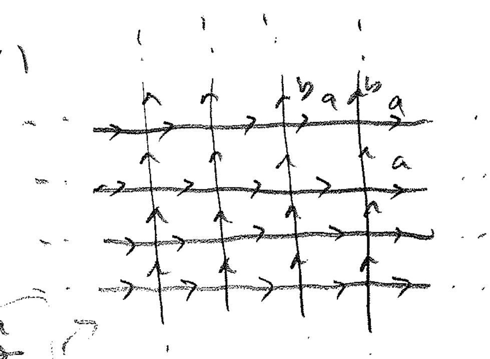
the following graphs are all covering spaces of $X$ w/ covering map that sends a-edges, b-edges homeomorphically onto a-edge & b-edge

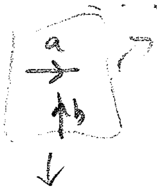
(i) 


(ii) 

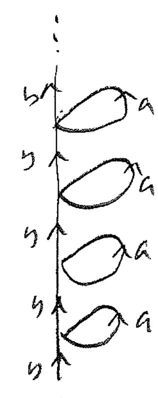

(iii) 

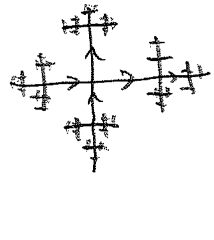

(iv) 


(v) 




$\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$

(vi) 


(vi) 

infinite regular 4-valent tree