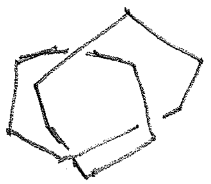
Math 428
1-20-11

[Comment: we're going to ignore some subtle topological issues to "get to some good stuff".] (5)

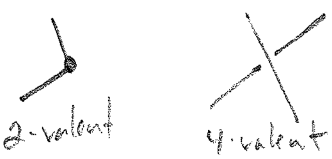
Last time we defined a knot in $\mathbb{R}^3$ to be a polygonal embedding $K: S^1 \to \mathbb{R}^3$ (or its image $K = K(S^1)$).

A nice projection exists and gives us a diagram of the knot



D

Can view D as a graph with vertices of valence 2 (images of vertices of K) and vertices of valence 4 (the crossings), the latter marked w/ over/under.



Also defined equivalence relation $\sim$ on knots (w/ 3 different descriptions).

<u>Exercise I.1</u>: If K, K' have the same diagram, then $K \sim K'$.

What if the diagrams are different?

Lets look at ways the diagrams could be different:

<u>planar isotopy</u>:

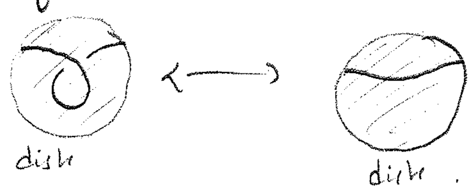
suppose K, K' have diagrams D, D' in a plane $\mathbb{R}^2$, say D, D' are related by <u>planar isotopy</u> if $\exists\ H: \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ w/ $H_t(x) = H(x,t)$, $H_t: \mathbb{R}^2 \to \mathbb{R}^2$ homeomorphism, $H_0 = $ identity on $\mathbb{R}^2$, $H_1(D) = D'$ respecting over/unders at crossings.

then $K \sim K'$: extend H (by identity on 3^{rd} coordinate in $\mathbb{R}^3$) to a map $\tilde{H}: \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3$, $\tilde{H}_1(K)$ = is a knot with the same projection at K', so $K \sim \tilde{H}_1(K) \sim K'$.

<u>Reidemeister moves</u> : There are 3 ways to modify the diagram of a knot that clearly does not change the knot type, those are called the Reidemeister moves ;

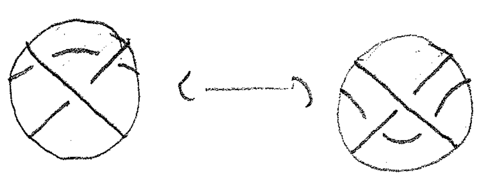
$D, D' \subset \mathbb{R}^2$ two diagrams, the same outside a disk ; differing in the disk by:

<u>type I</u> :

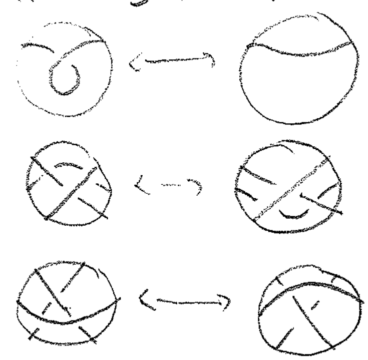


<u>type II</u> :

<u>type III</u> :



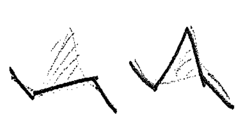
<u>Exercise I.2</u>: show that the "other Reidemeister moves" below are a consequence of these 3 (and planar isotopy)



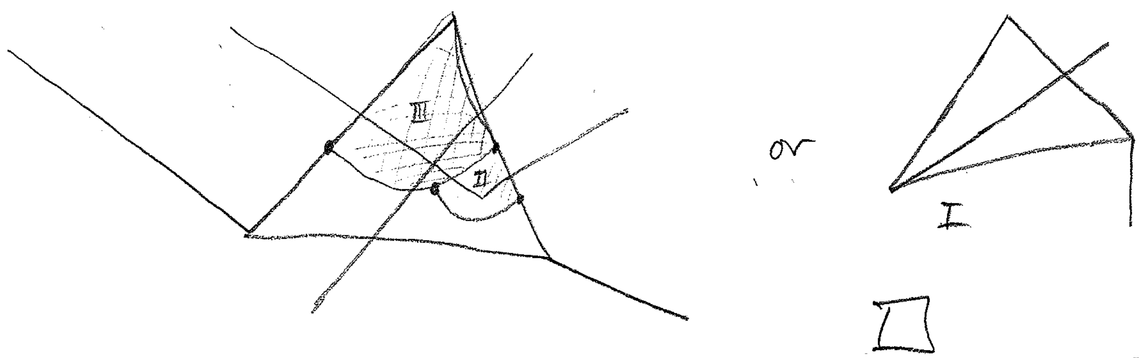
<u>Theorem I.3</u> K, K' two knots are equivalent iff the diagrams D, D' for K, K' differ by a finite sequence of Reidemeister moves I, II, III and planar isotopy.

<u>Idea of proof</u>: Need to see if $K \sim K'$, then D, D' differ as in theorem.

Know $\exists$ $K = K_1, K_2, \ldots, K_n = K'$ w/ K_i, K_{i+1} related by triangle moves



By small change in projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ we can assume π is a nice projection for every K_i. [Also note that small change of projection changes D & D' to planar isotopic diagrams]. Further, we can assume π is injective on all triangles defining moves. Now replace each triangle move with finitely many small moves of type I, II, and III. e.g.



[Remark: We can essentially take this theorem as defn of equivalence – i.e, $K \sim K'$ iff diagrams are related as in thm. This allows us to view knots and question of equiv. of knots as a combinatorial problem.]

How does this help? [replaced existence/non existence of one type of sequence with another].

We can use this to define knot invariants:

Let Knot($\mathbb{R}^3$) be the set of all knots in $\mathbb{R}^3$.

<u>Def'n</u> A <u>knot invariant</u> is a function

$$F : \text{Knot}(\mathbb{R}^3) \longrightarrow \Omega$$

where Ω is a set with an equivalence rel'n $\approx$ on it such that if $K \sim K'$, then $F(K) \approx F(K')$.

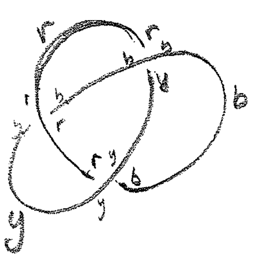
Ω could be a set of groups or abelian groups (w/ equiv. rel'n $\cong$), or polynomials, $\mathbb{R}, \mathbb{Z}, \mathbb{C}$, w/ trivial rel'n ... Anything!

How do we use Reid. moves to do this?

<u>Ex</u> $\text{Tri} : \text{Knot}(\mathbb{R}^3) \longrightarrow \mathbb{Z}$

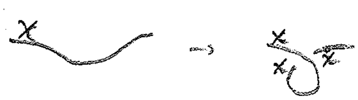
Tri(K) is defined as follows: Let D be a diagram of K

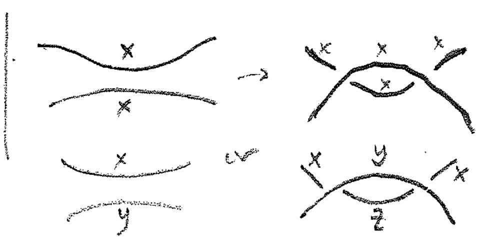
Tri(K) = # of ways to color overpassing arcs of D with 3 colors (using at least 2) so that at any crossing either all 3 colors appear, or exactly one does:



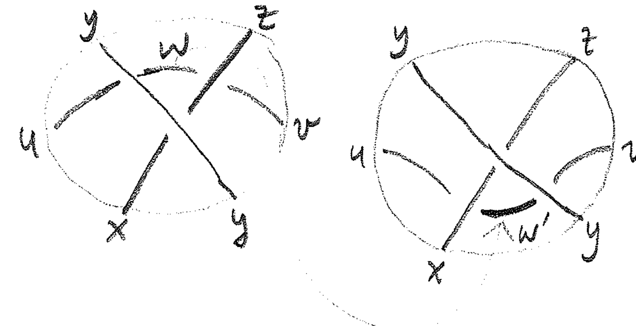
K = trefoil, Tri(K) = 6 (if two arcs are same color, all are)
all 3 arcs diff'nt color

If $K \sim K'$, why is Tri(K) = Tri(K')? check invariance under RI, II, III: that is, if D & D' differ by RI, II, III, then a coloring of one uniquely determines a coloring of other





III: obvious 1-1 correspondence between over arcs of D & D'



Same outside

check:

① w, w' are determined by u, v, x, y, z — easy ($u, y \Rightarrow w$)

② if u, v, x, y, z, w is a tricoloring, then $\exists\, w'$ s.t. u, v, x, y, z, w' is a tricoloring (it's unique by ①) — cases: • $x \neq y$, exactly 3 possibilities

②' reverse roles of w, w' • $x = y$, exactly 3 possibilities.

This proves

Theorem I.4 Tri is a knot invariant

Corollary I.5  $\not\simeq$ O

Links: {A link is a collection of pairwise disjoint knots}

Defn A link is a (polygonal) embedding $L: \coprod_{i=1}^{k} S^1 \to \mathbb{R}^3$. We also write $L = L(\coprod_{i=1}^{k} S^1)$. The restriction to one of the circles is called a component $L_1, \dots, L_k : S^1 \to \mathbb{R}^3$ (or $L_i(S^1) = L_i$) and each L_i is a knot. Say L has k-components in this case. [a one compt link is a knot]

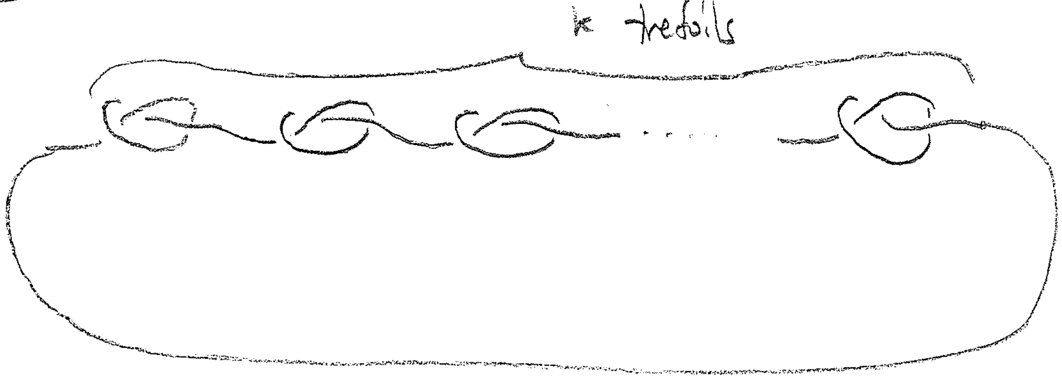
Same defn of equivalence, and I.1—I.4 also hold for links.

k-component unlink:

↑ move up after Corollary I.5

Corollary I.6: $\exists$ infinitely many, pairwise inequivalent knots.

proof: Consider the knot K_k:



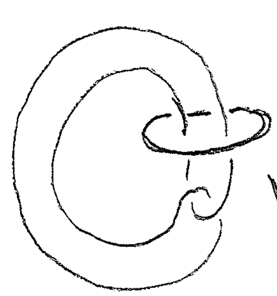
check that there are $3(3^k - 1)$ tricolorings, so

$$\mathrm{Tri}(K_k) = 3(3^k - 1).$$

Therefore, $\mathrm{Tri}(\mathrm{Knot}(\mathbb{R}^3))$ is infinite, hence so is the # of equivalence classes in $\mathrm{Knot}(\mathbb{R}^3)$.

Exercise I.3 There are only a countably infinite number of equivalence classes of knots. (of course, $\mathrm{Knot}(\mathbb{R}^3)$ is uncountable)

Link example:



W = Whitehead link

Note: $W \neq OO$ since $\mathrm{Tri}(W) = 0$, but $\mathrm{Tri}(OO) = 6$.