$$\gamma_1\gamma_2(t) = \begin{cases} \gamma_1(2t) & t \in [0,\tfrac{1}{2}] \\ \gamma_2(2t-1) & t \in [\tfrac{1}{2},1] \end{cases}$$ well defined since $\gamma_1(1) = x = \gamma_2(0)$

(continuous by Thm III.7.4)

More generally, if $\gamma_1, \gamma_2 : [0,1] \longrightarrow X$ are two paths w/ $\gamma_1(1) = \gamma_2(0)$, then the above defines a new path

$$\gamma_1\gamma_2 : [0,1] \longrightarrow X$$

Moreover, if we let $[\gamma]$ denote the homotopy class, rel endpts, of a path γ, then we define

$$[\gamma_1]\cdot[\gamma_2] = [\gamma_1\gamma_2]$$

provided $\gamma_1(1) = \gamma_2(0)$. — say γ_1, γ_2 are compatible

<u>Theorem III.35</u> This operation is well defined on homotopy classes, and

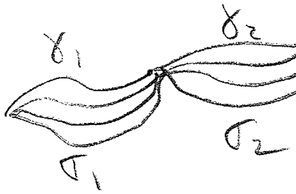
(1) $[\gamma_1]([\gamma_2][\gamma_3]) = ([\gamma_1][\gamma_2])[\gamma_3]$ $\quad \forall$ comp. paths $\gamma_1, \gamma_2, \gamma_3$

(2) If γ is a path, γ_0, γ_1 constant paths w/ $\gamma_0(t) = \gamma(0)$, $\gamma_1(t) = \gamma(1)$ $\forall t$, then $[\gamma][\gamma_1] = [\gamma_0][\gamma] = [\gamma]$.

(3) If γ is a path, γ_0, γ_1 as in (2) and $\bar{\gamma}(t) = \gamma(1-t)$, then

$$[\gamma][\bar{\gamma}] = [\gamma_0] \;,\; [\bar{\gamma}][\gamma] = [\gamma_1].$$

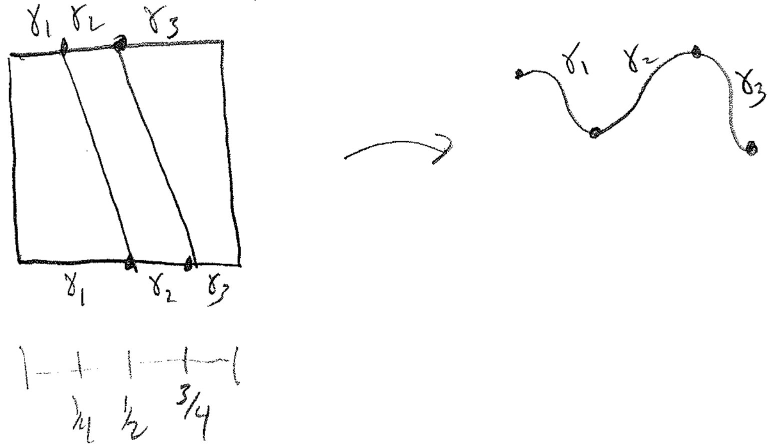
<u>proof</u> well definedness: $\gamma_i \simeq \sigma_i$ (rel endpts), $i = 1,2$. Let H_i be homotopies, $i=1,2$, then define $H_{12}(t,s) = \begin{cases} H_1(2t,s) & \forall s, \forall t \in [0,\tfrac{1}{2}] \\ H_2(2t-1,s) & \forall s, \forall t \in [\tfrac{1}{2},1] \end{cases}$



so $\gamma_1\gamma_2 \simeq \sigma_1\sigma_2$ (rel endpts), and well defined.

rest are straightforward exercises.

e.g. associativity (2), use picture to guide!



$\square$

Corollary III.36 Given a pointed space (X,x), the homotopy classes (rel endpts) of loops based at x:

$$\{[\gamma] \mid \gamma \text{ a loop based at } x\}$$

forms a group with operation $[\gamma][\sigma]$.

This group is called the <u>fundamental group of (X,x)</u> and is denoted $\pi_1(X,x)$. (1st homotopy group)

[of course, there are higher homotopy groups, but we will consider these]

Corollary III.37 If δ is a path in X w/ $\delta(0)=x$, $\delta(1)=y$, then $[\delta]_*: \pi_1(X,x) \longrightarrow \pi_1(X,y)$ defined by $[\delta]_* = [\bar\delta][\gamma][\delta]$ is an isomorphism that depends only on $[\delta]$.

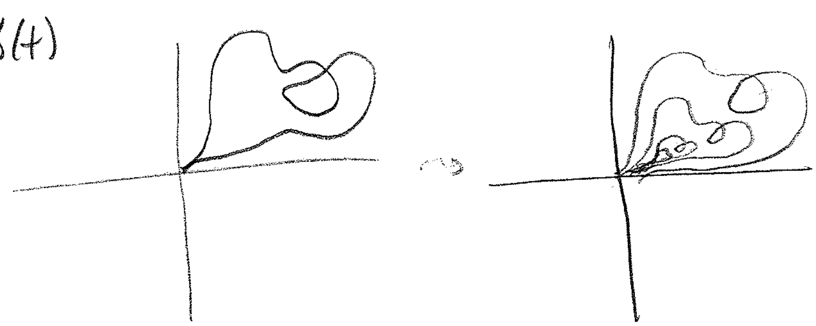
<u>Corollary III.38</u> If X is path connected then up to isomorphism, $\pi_1(X,x)$ is independent of x.

<u>Ex</u>: $\pi_1(\mathbb{R}^n, 0) \cong \{1\}\ \forall n.$ — say such a space is <u>simply connected</u>

<u>proof</u>: we need to show $[\gamma] = [c]$, where $c(t) = 0\ \forall t \in [0,1]$, for every loop γ based at 0. Indeed define a homotopy

$$H(t,s) = (1-s)\gamma(t)$$

"straight line homotopy"



$\forall$ — same construction shows any two maps into $\mathbb{R}^n$ are homotopic

$f, g : X \to \mathbb{R}^n$ set

$$H(x,t) = (1-t)f(x) + t\,g(x)$$ (similarly for convex subsets in $\mathbb{R}^n$)

— If $f(a) = g(a)\ \forall a \in A$, then homotopy is rel A. —

<u>Ex</u> $\pi_1(S^n, N) \cong \{1\}\ \forall n \geq 2$ $(N = (0,0,\dots,0,1)$, say$)$

<u>Proof</u>: $\gamma : [0,1] \to S^n$ a path.

γ is <u>uniformly continuous</u>; Given $\varepsilon > 0\ \exists \delta > 0$

so $\forall x, y \in [0,1]$ $|x-y| < \delta$, $|\gamma(x) - \gamma(y)| < \varepsilon$

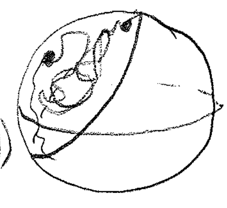
(follows since $[0,1]$ is compact)

Let $\varepsilon = 1$, and $\delta > 0$ be as above, set $k \in \mathbb{Z}_+$ so

$\frac{1}{k} < \delta$.

then $\forall\ j \in \{0, \dots, k-1\}$, $\gamma([j/k, j+1/k]) \subset B_1(\gamma(j/k))$

$B_1(\gamma(j/k))$



perform "straight line homotopy" to a nice arc

$$H(t,s) = \frac{(1-s)\gamma(t) + s\left(k\left(\frac{j+1}{k} - t\right)\gamma\left(\frac{j}{k}\right) + k\left(t - \frac{j}{k}\right)\gamma\left(\frac{j+1}{k}\right)\right)}{\left|(1-s)\gamma(t) + s\left(k\left(\frac{j+1}{k} - t\right)\gamma\left(\frac{j}{k}\right)\right) + k\left(t - \frac{j}{k}\right)\gamma\left(\frac{j+1}{k}\right)\right|}$$

for $t \in [j/k, j+1/k]$

Let γ' be resulting homotopic loop, $\gamma \simeq \gamma'$, (rel end pts)

observe that γ' is <u>not</u> surjective (it lies in intersection of S^n w/ k 2-planes), so $\gamma'([0,1]) \subset S^n - \{x\}$, some x. But $S^n - \{x\} \cong \mathbb{R}^n$, so γ' is htpc to constant map, hence $[\gamma] = [\gamma'] = [c]$ and $\pi_1(S^n, N) = \{1\}$.

The homeomorphism $f: S^n \setminus \{x\} \xrightarrow{\cong} \mathbb{R}^n$ allows us to transport the homotopy in $\mathbb{R}^n$ to one in $S^n \setminus \{x\}$; $H: [0,1] \times [0,1] \to \mathbb{R}^n$, then $f \circ H: [0,1] \times [0,1] \to \mathbb{R}^n$. More generally, we have, for any continuous map $f: X \to Y$, and $x \in X$, $y = f(x)$, an induced map

$f_*: \pi_1(X, x) \to \pi_1(Y, y)$

given by $f_*([\gamma]) = [f \circ \gamma]$ well defined since a homotopy H from γ to γ' induces $f \circ H$, a homotopy from $f \circ \gamma$ to $f \circ \gamma'$, In fact we have:

<u>Proposition III.39</u> Given $f: X \to Y$, continuous $f_*: \pi_1(X,x) \to \pi_1(Y, f(x))$ is a homomorphism. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are continuous, then $(g \circ f)_* = g_* \circ f_*$.

<u>proof</u> exercise $\square$.

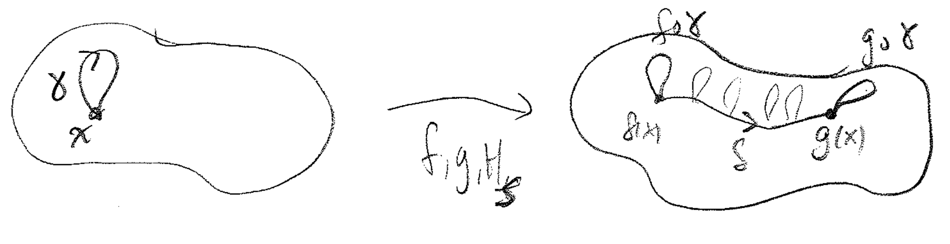
<u>Corollary III 40</u> If $f: X \to Y$ is a homeomorphism, then f_* is an isomorphism.

There is a coarser relation than homeomorphism for which implies f_* is an isomorphism. For this, we first note that if $f, g: X \to Y$ and H is a homotopy, $H_0 = f$, $H_1 = g$, then setting $\delta(t) = H(x,t)$ (here $x \in X$ is a basepoint), we have that δ is a path w/ $\delta(0) = f(x)$ and $\delta(1) = g(x)$.

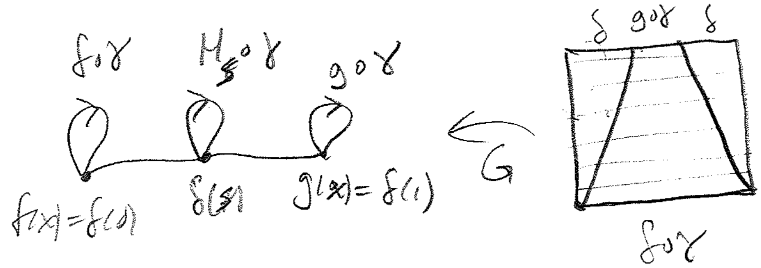
<u>Proposition III.41</u> With notation as above we have

$$[\delta]_* \circ f_* = g_* : \pi_1(X,x) \longrightarrow \pi_1(Y, g(x)).$$

<u>proof</u> equiv. formulation is perhaps simpler: $f_* = [\bar{\delta}]_* \circ g_*$



need to see that $f \circ \gamma \simeq \delta (g \circ \gamma) \bar{\delta}$ rel endpts picture is more useful than formula



$$G(t,s) = \begin{cases} \delta(3t) & 0 \le t \le s/3 \\ H\left(\gamma\left(\frac{3}{3-2s}\left(t-\frac{s}{3}\right)\right), s\right) & s/3 \le t \le 1 - s/3 \\ \delta(3-3t) & 1-s/3 \le t \le 1 \end{cases}$$

$G_s(t)$: $G_0(t) = f\circ\gamma(t)$, $G_1(t) = \delta(g\circ\gamma)\bar\delta(t)$ $\square$

<u>Corollary III.42</u> If $f, g : (X,x) \to (Y,y)$ are homotopic (rel x) then $f_* = g_*$.

<u>Defn</u> A <u>homotopy equivalence</u> is a cont. map $f: X \to Y$ such that $\exists$ cont. map $g: Y \to X$ w/ $f\circ g \simeq id_X$ and $g\circ f \simeq id_Y$. g is called a <u>homotopy inverse</u> for f.

<u>Corollary III.43</u> If $f: X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X,x) \to \pi_1(Y, f(x))$ is an isomorphism

<u>Exercise</u>: homotopy equivalence is an equiv. reln on top. spaces.