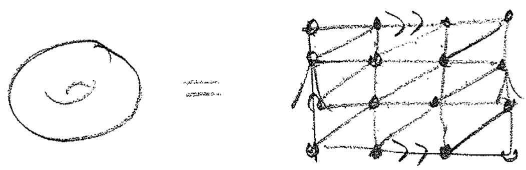
Math 428 4/5/11

Last time -

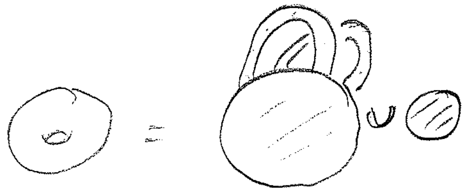
Euler characteristic of a surface :

- $\chi(S) = V - E + F$ for a triangulation.



$\chi = 9 - 27 + 18 = 0.$

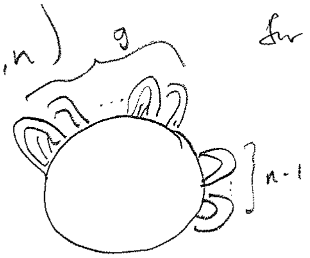
- $\chi(S) = \#(\text{0-handles}) - \#(\text{1-handles}) + \#(\text{2-handles})$ for a handle decomposition



$\chi = 1 - 2 + 1 = 0.$

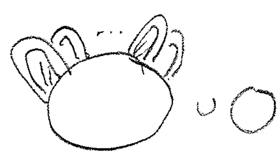
Theorem III.32 If S is a compact surface-with-boundary then $\chi(S)$ is independent of the triangulation or handle decomposition used to compute it (and both methods give same value).

Ex: $\chi(S_{g,n})$ for $n \geq 1$:



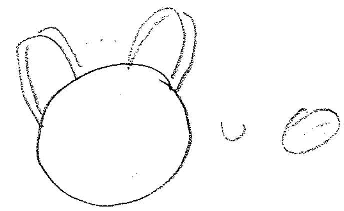
$$\chi(S_{g,n}) = 1 - 2g - (n-1)$$
$$= 2 - 2g - n.$$

$$\chi(S_g) = \chi(S_{g,0}) = 1 - 2g + 1 = 2 - 2g$$



$k \geq 1$

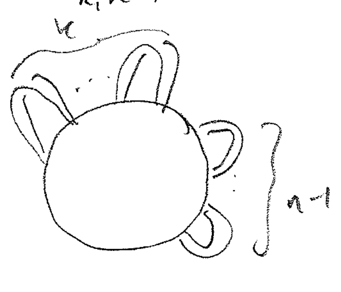
$$\chi(M_k) = \chi(M_{k,0}) = 1-k+1 = 2-k$$



$$\Rightarrow \begin{cases} \chi(\mathbb{RP}^2) = 1 \\ \chi(\text{Klein bottle}) = 0 \end{cases}$$

$k \geq 1, \; n \geq 1$

$$\chi(M_{k,n}) = 1-k-(n-1) = 2-k-n$$



Proposition III.33

$\chi(S_{g,n}) = 2-2g-n \quad \forall\, g,n \geq 0$

$\chi(M_{k,n}) = 2-k-n \quad \forall\, k \geq 1,\, n \geq 0.$

Corollary III.34 (2nd half classification) All surfaces

$\{S_{g,n}\}_{g,n\geq 0} \cup \{M_{k,n}\}_{k\geq 1, n \geq 0}$ are all pairwise non-homeomorphic.

Proof Last time we saw that $S_{g,n} \not\cong M_{k,m}$ for all $g,n,m \geq 0,\, k \geq 1$ (orientability is a homeomorphism invariant). Just need to check that

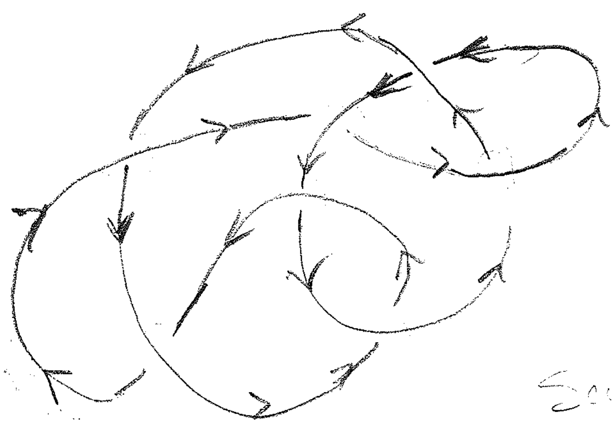
$S_{g,n} \cong S_{g',n'} \Rightarrow g=g'$ & $n=n'$ and $M_{k,n} \cong M_{k',n'} \Rightarrow k=k', n=n'$.

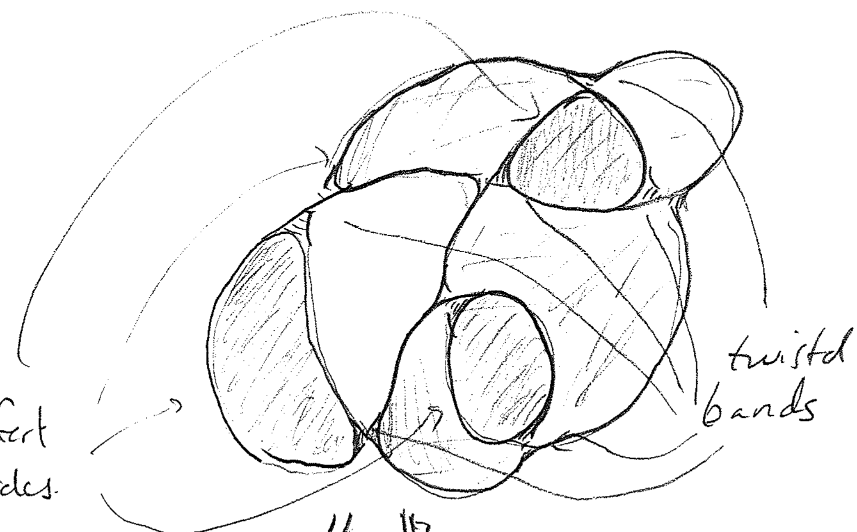
For this, observe that # boundary components is preserved by homeomorphism, so $n=n'$ then proposition $\Rightarrow g=g'$ & $k=k'$ respectively □

So, surface is determined by

(1) orientability, (2) # boundary compts, & (3) χ.

Seifert's algorithm





$S \cong S_{g,1}$ — what is g?

natural handle decompositition ⟹

$$\chi(S) = \#\text{ Seifert circles} - \#\text{ bands}$$
$$= \#\text{ Seifert circles} - \underline{\#\text{ crossings}} = s - c$$

$$\chi(S) = 4 - 7 = -3 = 2 - 2g - 1 = 1 - 2g$$

$$-4 = -2g$$
$$g = 2$$

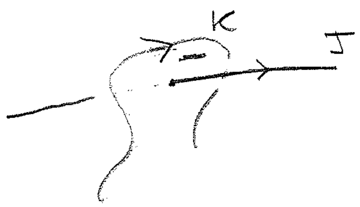
$$1 - 2g = s - c \implies g = \frac{c - s + 1}{2}$$

linking number revisited:

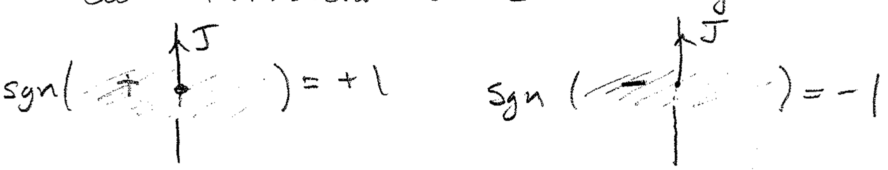
recall $lk(J,K) = \#\left(\text{—}\!\!\downarrow_K\!\!\to J\right) - \#\left(\text{—}\!\!\uparrow_K\!\!\to J\right)$

Let S be the surface obtained by apply Seifert's algorithm to K. Orient S by choosing $+/-$ sides so that the boundary is oriented right-to-left on $+$-side

By small perturbation we can assume $J \cap S$ is finite, and not tangent



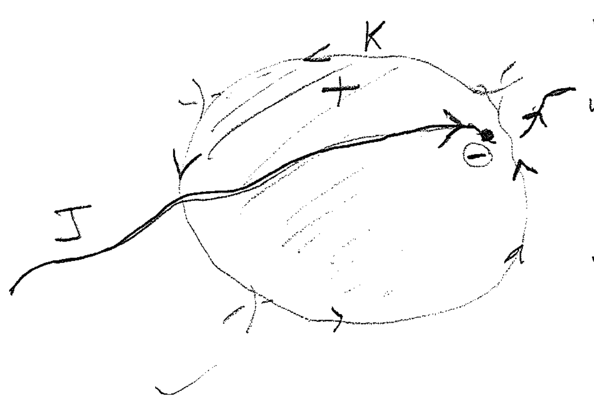
count intersection $J \cap S$ with sign so



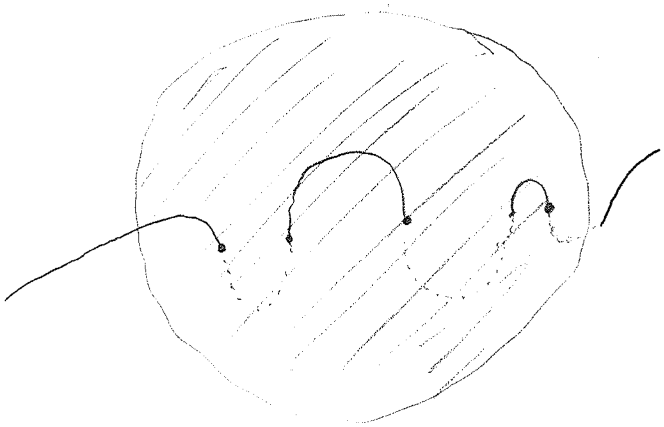
$$\langle J, S\rangle = \sum_{x \in J\cap S} \mathrm{sgn}(x)$$

Compute: $\langle J, S\rangle = lk(J,K)$

to see this, can assume that J only intersects disks bounded by Seifert circles (not strips). These are parallel to projection plane.



View J as high above S, except when it passes under K, then easy to see. Actual J may meet S more, but all cancel in pairs.

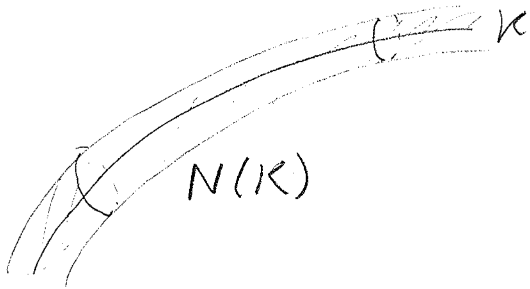


later we will see that this is true for any seifert surface (not just the one from Seiferts algorithm)

Recall: Exterior of K in S^3 is

$$X(K) = S^3 \setminus N(K)$$

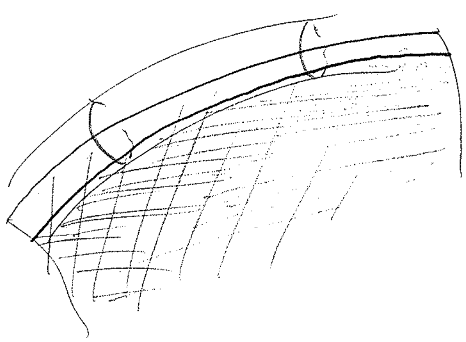
where N is a small open nbhd of K

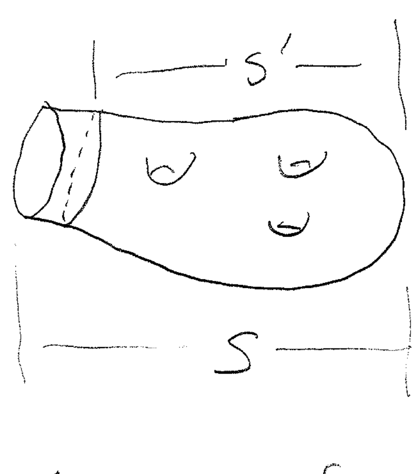


$X(K)$ compact 3-mfd w/ boundary, $\partial X(K) \cong T^2$.

A seifert surface S for K intersects $X(K)$ in a surface $\cong S$:

$S' := S \cap X(K)$ has $S' \cong S$.





Convenient to sometimes think of Seifert surface as a surface in $X(K)$, this is what we mean.

Fundamental group

compactness, connectedness, # compt, path components are all invariants of top. spaces (i.e. same for homeomorphic spaces), but they don't distinguish many spaces.

Q What invariants can be used to distinguish cpct. orient. surfaces?

A: Euler characteristic, ... what else?

We want to define a group associated to a top. space so homeomorphic spaces have isomorphic groups (actually slightly more restricted...). 1st defns,

Defn A homotopy is a continuous 1-parameter family of continuous maps. More precisely, ~~for~~ top. spaces X & Y it is a continuous map

$$H: X \times [0,1] \longrightarrow Y$$

we sometimes write $H_t(x) = H(x,t)$, so $\forall\, t \in [0,1]$, $H_t : X \to Y$.

If $f : X \to Y$ is a continuous map, a homotopy of f is a homotopy

$$H: X \times [0,1] \to Y$$

so that $H_0 = f$. Two maps f and g are homotopic if $\exists$ a homotopy $H : X \times [0,1] \to Y$ s.t. $H_0 = f$, $H_1 = g$. Write $f \simeq g$

<u>Exercise</u>: $\simeq$ is an equiv. rel'n on continuous maps from X to Y.

<u>Rk</u>: previous notion of isotopy is a special case.

If $A \subset X$, a homotopy $H: X \times [0,1] \to Y$ is said to be <u>relative to A</u> (<u>rel A</u>) if $H(a,t) = H(a,0)$ $\forall\, t \in [0,1]$, $a \in A$.

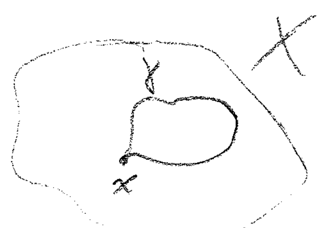
– it doesn't move A.

<u>Ex</u> A homotopy of a path rel end pts:

A <u>pointed top space</u> is a top. space X w/ a base point $x \in X$: (X,x). A <u>loop based at x in X</u> is a path (cont map)

$\gamma: [0,1] \to X$

w/ $\gamma(0) = \gamma(1) = x$



Given two loops $\gamma_1, \gamma_2: [0,1] \to X$ based at x, define their <u>product</u> to be $\gamma_1\gamma_2: [0,1] \to X$ given by