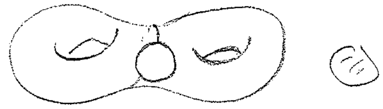
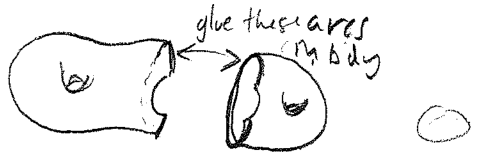
 ;

work backward: remove a disk:



cut along an arc



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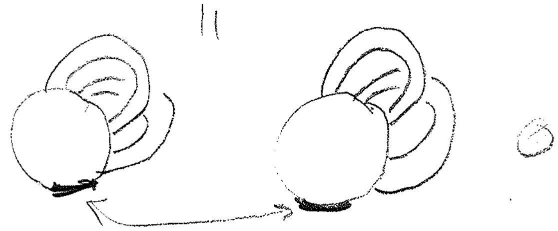

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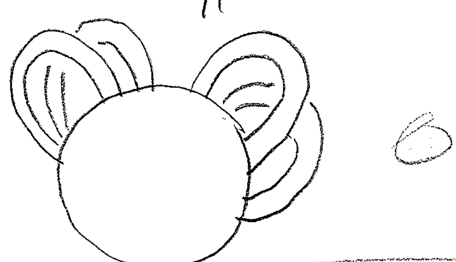
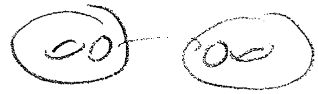


Get  from

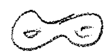
remove disks & glue



This is called <u>connected sum</u>:

the surface obtained from a pair of surfaces S, S' by removing disks and gluing.

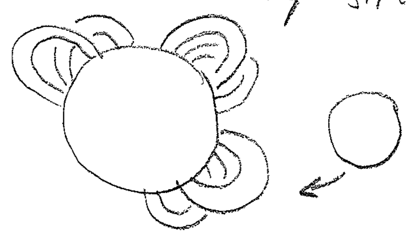
write $S \# S'$.

So  $= T^2 \# T^2$.

and

 $= T^2 \# T^2 \# T^2$

handle decomposition



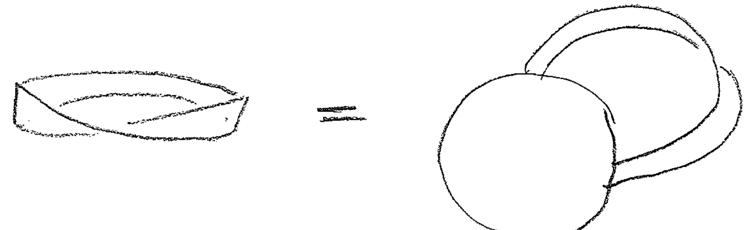
More generally

$\underbrace{\hspace{6cm}}_{g}$

=

$\underbrace{T^2 \# T^2 \# \dots \# T^2}_{g \text{ times}} = S_g =$ <u>compact orientable surface of genus g</u>

Ex Möbius band: $M =$



$\partial M \cong S^1$ (just one component)

What if we glue on a disk to this boundary component?

$=: \mathbb{RP}^2$

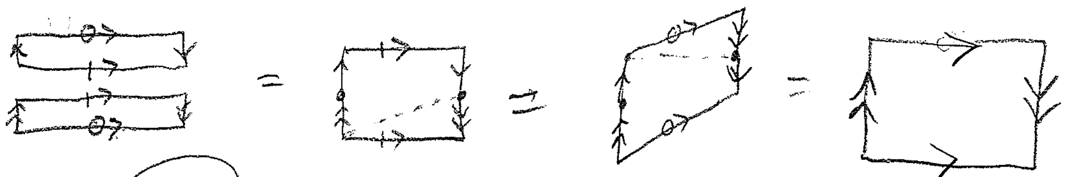
alternatively, $\mathbb{RP}^2 = S^2/\sim$ where $x \sim -x$ $\forall x \in S^2 \subset \mathbb{R}^3$

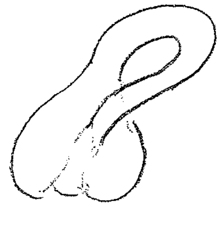
we identify antipodal points.

Exercise III.15 Explain why $\mathbb{RP}^2 \cong M \cup D$ as described above.

Hint: $\pi : S^2 \longrightarrow \mathbb{RP}^2$, and look at upper hemisphere.

Let $M_k = \underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{k}$, $M_2 = \mathbb{RP}^2 \# \mathbb{RP}^2$ is called a Klein bottle: $\mathbb{RP}^2 \# \mathbb{RP}^2 =$ two Möbius bands glued along their boundaries:





Theorem III.30 Any compact surface (without boundary) is homeomorphic to exactly one of $S_0, S_1, S_2, \ldots$ or $M_1, M_2, \ldots$

[In particular, these are pairwise nonhomeomorphic]

A compact surface with boundary is homeomorphic to one of these, with some # of pairwise disjoint disks removed.

Can write $S_{g,b} = S_g \setminus b$-disks

$M_{k,b} = M_k \setminus b$-disks

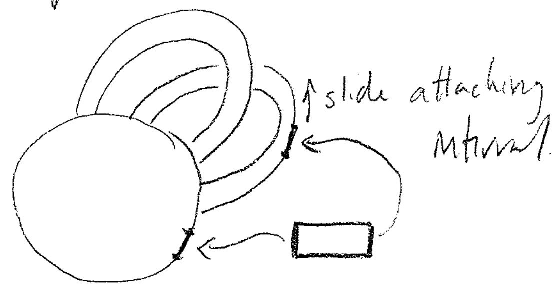
Note $S_{g,b}$, $M_{k,b}$ each have b boundary components.

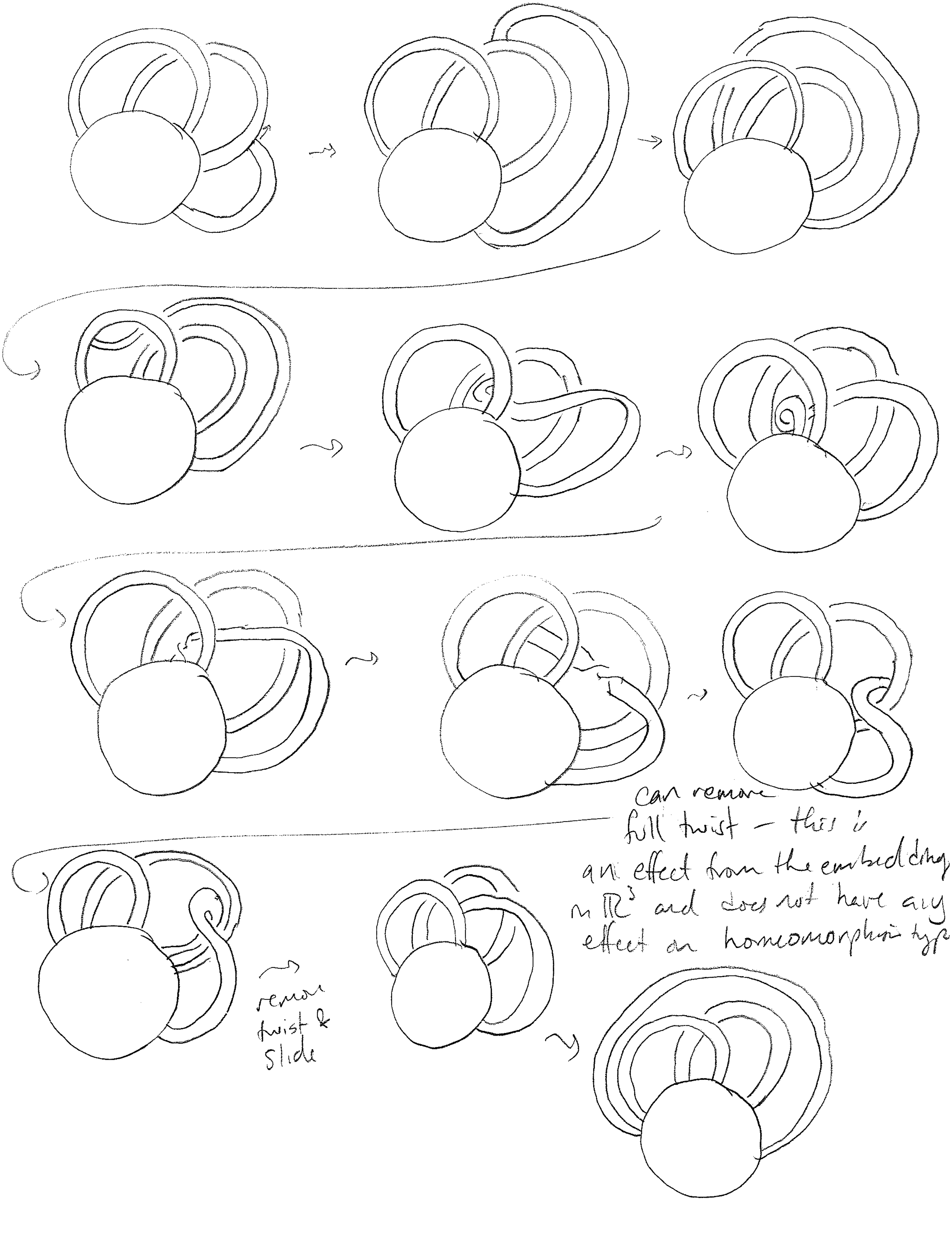
$S_{g,0} = S_g$, $M_{k,0} = M_k$.

We do $\frac{1}{2}$ of the proof now (any surface is homeomorphic to one on the list) and the other ½ later - requires more tools.

Need to describe operations on handle decompositions that do not change homeomorphism type

Main operation: handle slides.





remove twist & slide

can remove full twist – this is an effect from the embedding in $\mathbb{R}^3$ and does not have any effect on homeomorphism type

Formally: Attaching $[0,1]\times[0,1]$ to X_n along ∂X_n requires a gluing embedding: $\varphi_{n+1}: [0,1]\times\{0,1\} \longrightarrow \partial X_n$

and $X_{n+1} = X_n \cup [0,1]\times[0,1] \big/ \sim$ where $\varphi_{n+1}(x) \sim x$, $\forall x \in [0,1]\times\{0,1\}$

$=: X_n \cup_{\varphi_{n+1}} [0,1]\times[0,1]$.

A handle slide is family of gluing embeddings that obtained by composing φ_{n+1} with an isotopy:

$H: X_n \times [0,1] \longrightarrow X_n$ continuous s.t.

$H(x,t) = H_t(x)$ has $H_t : X_n \longrightarrow X_n$ a homeomorphism $\forall t \in [0,1]$

& $H_0(x) = x$ $\forall x \in X_n$.

so $H_t \circ \varphi_n$ "slides the attaching map" as t varies.

Observe that we can use H_t to construct the required homeo

$F_t : X_n \cup_{\varphi_{n+1}} [0,1]\times[0,1] \xrightarrow{\cong} X_n \cup_{H_t \circ \varphi_{n+1}} [0,1]\times[0,1]$ $\forall t$ by

$F_t(x) = H_t(x)$ $\forall x \in X_n$ & identity on $[0,1]\times[0,1]$.

[Imagine H_t is the identity outside a nbhd of $\partial X_n \cong \partial X_n \times [0,1)$]

Also — can check that any two ways of attaching a disk to a circle boundary component — gluing entire boundary of disk to a component of the boundary of the surface-with-boundary — results in homeomorphic surfaces — easier: Need to define a homeo:

$F: X \cup_f D \longrightarrow X \cup_g D$ where $f, g : \partial D \xrightarrow{\cong} \partial_0 X$ and $\partial_0 X$ is a component of ∂X; HINT: set $F(x) = x$ $\forall x \in X$ and use polar coordinates on D