Proposition III.27 X^* is a compact surface (w/o boundary)

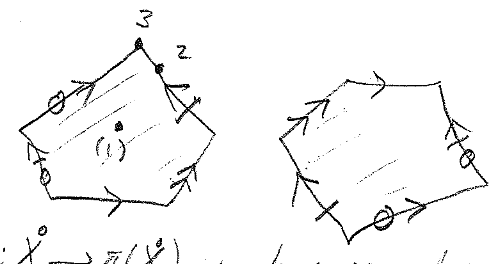
proof 1st we verify that X^* is locally Euclidean. Let $\pi : X \to X^*$.

There are 3 types of points in X^*:

(1) points in $\pi(\mathring{X})$

(2) points in $\pi(\mathring{S}_i)$, some side S_i

(3) points in $\pi(v)$, v a vertex

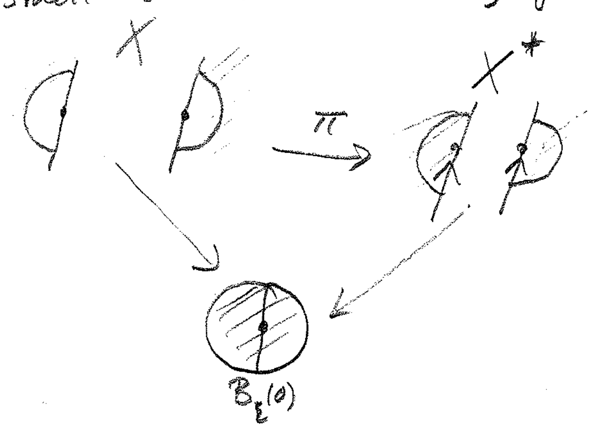


① note $\pi(\mathring{X})$ is an open subset of X^* and $\pi : \mathring{X} \to \pi(\mathring{X})$ is a homeomorphism (In particular $\pi|_{\mathring{X}}$ is a bijection)

this gives the desired nbhd $\forall\, \bar{x} \in \pi(\mathring{X})$

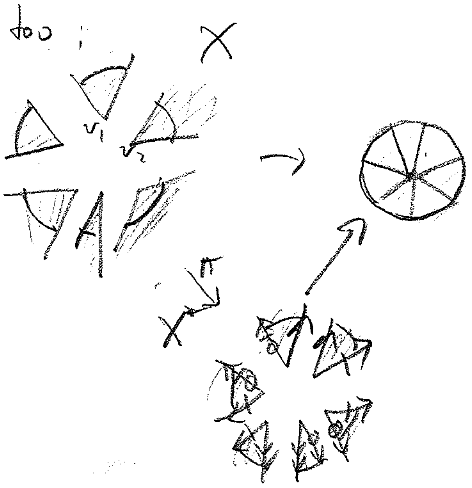
② In this case $\bar{x} \in \pi(\mathring{S}_i)$, let $\phi_{ij} : S_i \to S_j$, then $\bar{x} = \{x, y\}$, w/ $y = \phi_{ij} x$, $x \in S_i$.

2 small ½ disks around x & y give desired nbhd for $\bar{x}$:



(apply Prop III.25)

③ We have $\pi^{-1}(\pi(v)) = v_1, \ldots, v_k$ union of vertices, construct appropriate nbhd here, too:

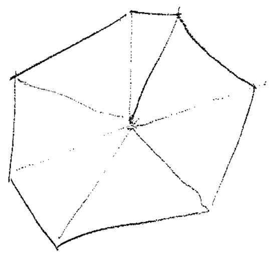


(apply Prop III.25)

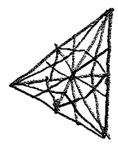
X^* compact since X is and $\pi : X \to X^*$ is cont.

We can check that X^* is 2^{nd} countable & Hausdorff, but also follows by continuously injecting it into $\mathbb{R}^n$, some (large) n as follows

1^{st}, subdivide each polygon into triangles.



Perform the $\underline{2^{nd}\ \text{barycentric subdivision}}$ on each triangle.



The restriction of π to each triangle is injective and if two triangles Δ_1, Δ_2 have $\pi(\Delta_1) \cap \pi(\Delta_2) \neq \emptyset$, then they intersect in (the image of) a single vertex, or single edge, or $\pi(\Delta_1) = \pi(\Delta_2)$

$\Rightarrow$ each triangle $\pi(\Delta)$ is determined by its 3 vertices.

Let $v_0, \dots, v_n$ be the set of all vertices. Define a map

$\bar{f} : X^* \longrightarrow \mathbb{R}^n$ by sending $v_0 = 0$, $v_i = e_i$ $\left(= (0, 0, \dots, 0, \underset{i}{1}, 0, \dots 0) \right)$ $i = 1 \dots n$ and extending linearly over triangles.

the corresponding map $f : X \longrightarrow \mathbb{R}^n$ is obviously continuous, so $\bar{f} : X^* \longrightarrow \mathbb{R}^n$ is.

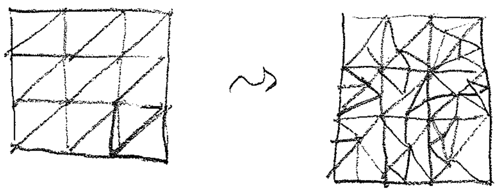
each triangle is mapped to a distinct 2-d hyperplane, easy to check $\bar{f}$ is injective. X compact $\Rightarrow$ X^* is an embedding.

So this means that X^* is Hausdorff & 2^{nd} countable in $\mathbb{R}^n$ is $\square$.

Defn A triangulation of a compact surface-with-bdry S is a homeomorphism $f : S \longrightarrow Z \subset \mathbb{R}^n$, where Z is a union of linearly embedded triangles in $\mathbb{R}^n$, which pairwise intersect in either an entire (single) edge, a vertex or not at all.

Observe that Z is completely determined by its vertices $v_0, \dots, v_n$ the data of which triples $\{v_i, v_j, v_k\}$ are vertices of a triangle in Z.

Theorem III.28 Every compact surface with boundary admits a triangulation. Any two triangulations differ by subdivision.



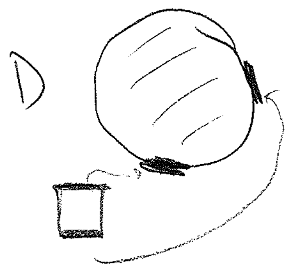
[also true for any surface w/ bdy, not just compact — need ∞ly many triangles, in general though]

Proof is quite hard we'll skip it.

Another way to describe surfaces is by <u>handle decompositions</u>:

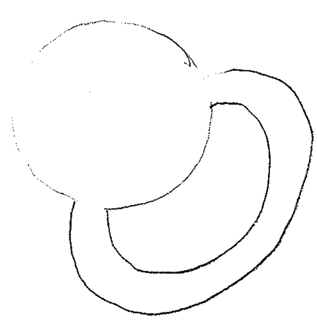
Here's the reverse construction:

Start with a disk $D = X_0$ (a compact surface with boundary)

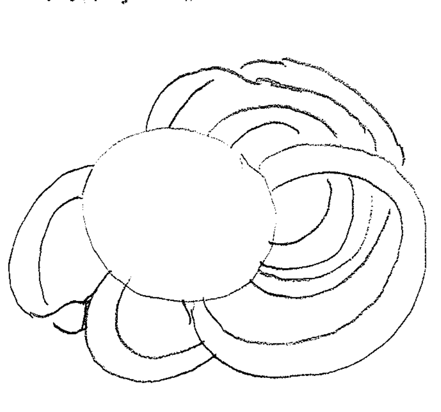


we attach a 1-handle, $[0,1]\times[0,1]$, by gluing $[0,1]\times\{0\}$ and $[0,1]\times\{1\}$ to disjoint arcs in ∂D

— this is still a compact surface with boundary X_1



Continue to add as many (finitely many) 1-handles to the boundary of the surface w/ boundary constructed thus ~~far~~



$X_0 \rightsquigarrow X_1 \rightsquigarrow X_2 \rightsquigarrow \dots \rightsquigarrow X_n$

Finally, we can glue a disk D (2-handles) along its boundary to a boundary circle of X_n to produce X_{n+1} — still a surface with boundary. repeat as many times as desired (no more than the # of boundary components of X_n)

[note, each boundary component of X_j is easily seen to be homeomorphic to a disjoint union of finitely many circles since its built from arcs — true for any compact mfd]

A handle decomposition of a compact surface with boundary S is a homeomorphism from S to a surface built from this construction.

Theorem III. 29 Any connected compact surface with boundary admits a handle decomposition

this follows from Thm III. 28 (in fact it is equivalent).

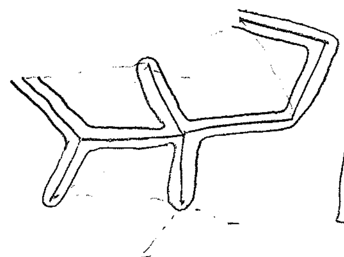
Sketch of proof (for compact surface S w/o boundary).

① The union of edges and vertices of triangulation is a graph Γ.

Let $T \subset \Gamma$ be a maximal tree (a connected subgraph with no cycles (embedded circles) that visits every vertex)

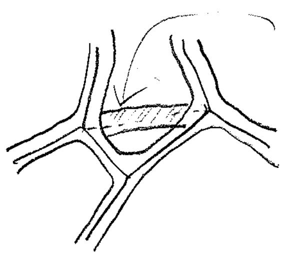
Exercise III.14 A maximal tree exists in any finite connected graph (in fact, it exists in any connected graph).

(2) a small nbhd of the tree is a disk:

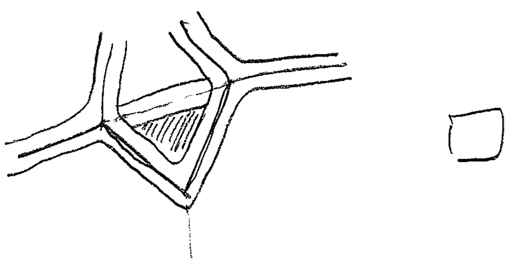


[can do this formally with a particular way of defining the nbhd — these are just unions of triangles in some $\mathbb{R}^n$]

(3) small nbhd of each remaining edge is an attached 1-handle

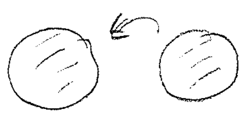


(4) all that remain are disks in triangles — these are the attached disks



EX: handle decomp. for S^2, T^2, 

S^2: no 1-handles:



T^2: