A slight generalization

Def'n upper ½-space $H^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$

the boundary of H^n is $\partial H^n = \{(x_1,\dots,x_{n-1},0) \in \mathbb{R}^n\}$

· A 2nd countable Hausdorff space M is an <u>n-manifold-with-boundary</u> if $\forall\, m \in M$, $\exists$ a nbhd U of m and a homeo $f: U \to V \underset{\text{open}}{\subseteq} H^n$.

The <u>boundary of M</u> is the subset

$$\partial M = \{ m \in M \mid \exists \text{ loc. hom } f: U \to V \underset{\text{open}}{\subseteq} H^n \text{ w/ } f(m) \in \partial H^n\}$$

<u>Fact</u>: If M is an n-mfd-w/boundary, $f_i: U \to V_i \subset H^n$ $i=1,2$ homeos $f_2 \circ f_1^{-1}: V_1 \to V_2$ takes $V_1 \cap \partial H^n$ onto $V_2 \cap \partial H^n$ — so ∂M is well-defined indep. of choice of $f: U \to V$.

<u>EX</u> · Any n-mfd M is an n-mfd w/ boundary, $\partial M = \emptyset$

· H^n

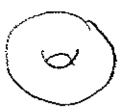
· open subsets

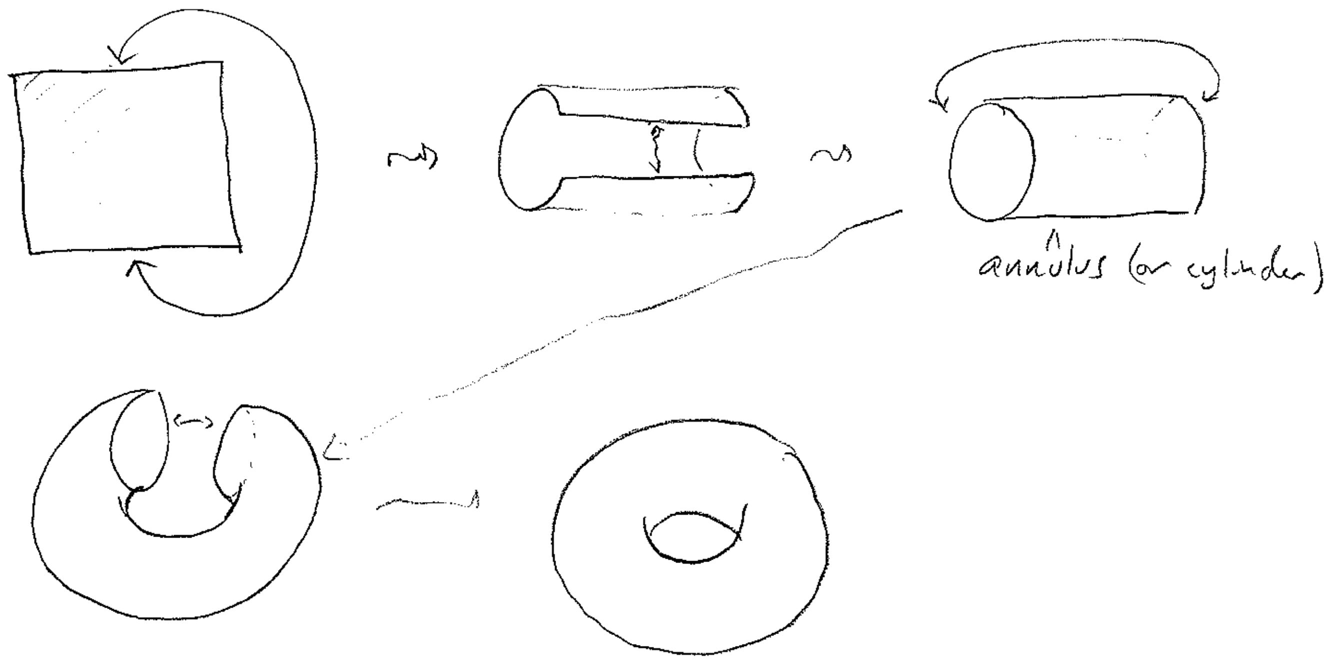
· $\bar{B}^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, $\partial \bar{B}^n = S^{n-1}$

— observe that by Fact, M an n-mfd w/ bdy, ∂M is an $(n-1)$-mfd —

Def'n A <u>surface</u> is a 2-mfd and a <u>surface-with-boundary</u> is a 2-mfd w/ bdry

* sometimes we refer to either one as a surface *.

<u>EX</u> S^2, $\mathbb{R}^2$, torus T^2 =  can describe T^2 by "gluing sides of a square"



To make this construction precise, we use use notion of quotient space.

Defn If X, Y are top. spaces, then a surjective map $f: X \to Y$ is a quotient map if a set $U \subset Y$ is open if and only if $f^{-1}(U)$ is open

- so f is continuous, but this is more.
- equiv: A is closed iff $f^{-1}(A)$ is closed.

Observe: A bijection $f: X \to Y$ is a quotient map iff f is a homeomorphism.

If X compact, Y Hausdorff, then $f: X \to Y$ surjective is a quotient map iff f is continuous. EX $\pi_X: X \times Y \to X$, $\pi_Y: X \times Y \to Y$ are quotient maps.

Proposition III.25 Suppose $\pi: X \to Y$ is a quotient map and

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur, "\bar{f}"'] \arrow[u, phantom, shift left=4, "\circ"] & \end{tikzcd}$$

Then f is continuous iff $\bar{f}$ is continuous.

<u>proof</u>: We just note that for any subset $U \subset Z$,

$$f^{-1}(U) = (\bar{f}\pi)^{-1}(U) = \pi^{-1}(\bar{f}^{-1}(U)).$$

So $f^{-1}(U)$ is open iff $\bar{f}^{-1}(U)$ is open, so f is continuous iff $\bar{f}$ is continuous. □

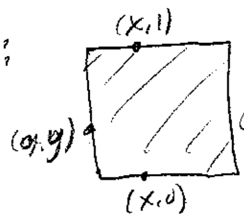
Given a top. space X and an equiv. relⁿ $\sim$ on X, let $X^* = X/\sim$ = the set of equivalence classes of elts of X and $\pi: X \to X^*$.

<u>Proposition III.26</u> $\exists!$ top on X^* so that π is a quotient map.

<u>proof</u>: just check that declaring $U \subset X^*$ to be open iff U is open defines a topology on X^*. □

This topology is called the <u>quotient topology</u> on X^*.

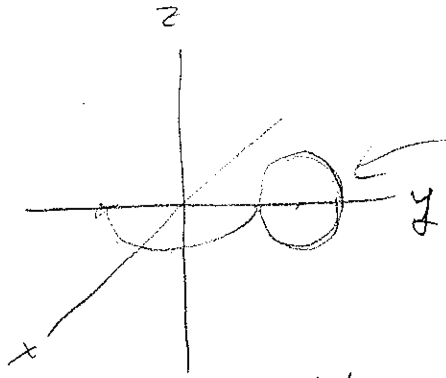
<u>Ex</u>:



$[0,1]\times[0,1] = X$, $X^* = X/\sim$ where $\sim$ is generated by $(x,0)\sim(x,1)$ $\forall x$, $(0,y)\sim(1,y)$ $\forall y$ (so $\sim$ is the smallest equiv. relⁿ containing these relations).

equivalence classes have $\begin{cases} 1 \text{ elt if in } (0,1)\times(0,1) \\ 2 \text{ elts if in sides, not corner} \\ 4 \text{ elts if in corner} \end{cases}$

Give X^* quotient topology. Claim $X^* \cong T^2$



T^2 = surface of revolution.

$(y-2)^2 + z^2 = 1$ on yz-plane
revolve around z axis.

circle: $\gamma(s) = (0, 2+\cos 2\pi s, \sin 2\pi s)$

Torus: $f(s,t) = \big((2+\cos(2\pi s))(\sin(2\pi t)),$
$(2+\cos(2\pi s))\cos(2\pi t), \sin 2\pi s\big)$

$f: X \to T^2$ continuous surjection, X Hausdorff

$\Rightarrow$ f quotient map. Observe $(s,t)\sim(s',t') \iff f(s,t) = f(s',t')$

so get

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circ"] & T^2 \\ X^* \arrow[ur, "\bar f"'] & \end{tikzcd}$$

$\bar f$ is continuous bijection by Prop. III. 25

and

$\bar f^{-1}$ is continuous also by Prop III.25 since f is a quotient map, too,

writing

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circ"] & T^2 \arrow[dl, "\bar f^{-1} = \bar\pi"] \\ X^* & \end{tikzcd}$$

So, $\bar f$ is a homeomorphism and $X^* \cong T^2$.

<u>EX</u> Gluing polygons to produce surfaces.
previous example gives general idea for constructing surfaces.

<u>Defn</u> Given a collection of pairwise disjoint compact polygons in the plane, $P_1, \ldots, P_n \subseteq \mathbb{R}^2$, a <u>side pairing</u> is a collection of linear (affine) homeomorphisms (*) between some of the sides $\Phi = \{\phi_{ij} : S_i \to S_j\}$ such that

① Given a side S_i, there is exactly one other side S_j such that $\exists (\phi_{ij} : S_i \to S_j) \in \Phi$. Moreover $S_j \neq S_i$ and if such ϕ_{ij} exists, there is only one such map.

② If $\phi_{ij} : S_i \to S_j \in \Phi$, then $\exists\, \phi_{ji} : S_j \to S_i \in \Phi$ and $\phi_{ij}^{-1} = \phi_{ji}$.

[We can assume if we like that the P_i's are convex polygons for simplicity]

So, in this case Φ "pairs up the sides", and we can glue them using Φ:

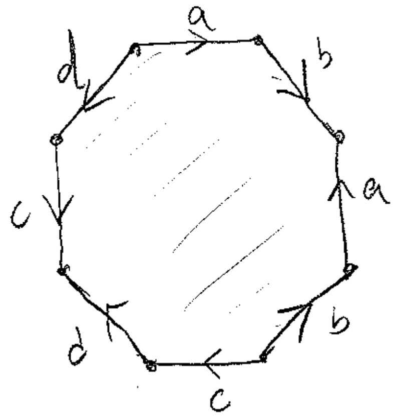
Let $X = P_1 \cup \ldots \cup P_n$, $X^* = X/\sim$ where $\sim$ is the equivalence relation generated by

$x \sim \phi_{ij}(x)$ $\forall (\phi_{ij} : S_i \to S_j) \in \Phi$ and $\forall x \in S_i$.

So, we glue together polygons according to Φ

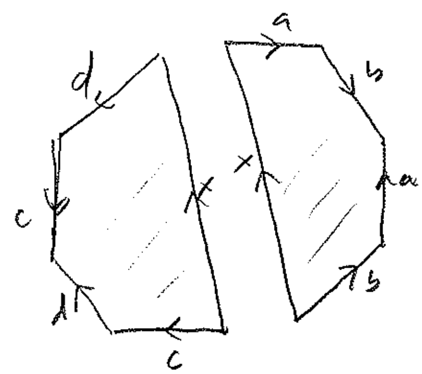
(*) So if sides are parametrized by $\gamma_i(t) = a_i t + b_i(1-t)$, $\gamma_j(t) = a_j t + b_j(1-t)$, then $\phi_{ij}(\gamma_i(t)) = \gamma_j(t)$ or $= \gamma_j(1-t)$ depending

EX

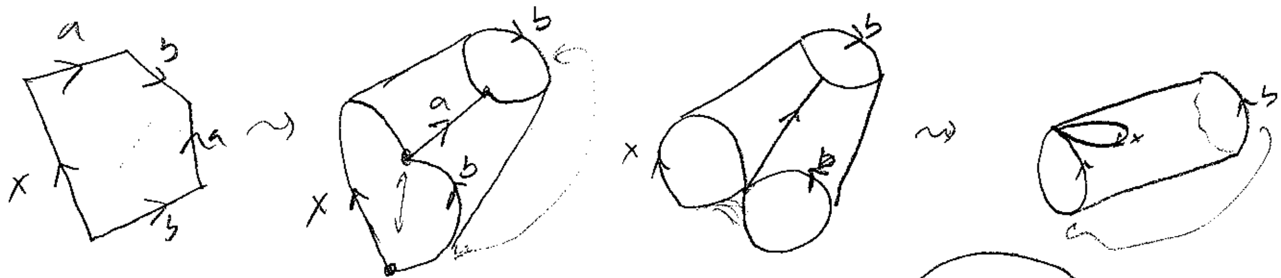


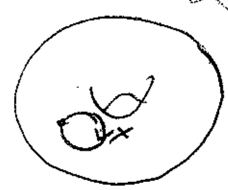
$\rightsquigarrow X^*$ What does X^* look like?

Same as

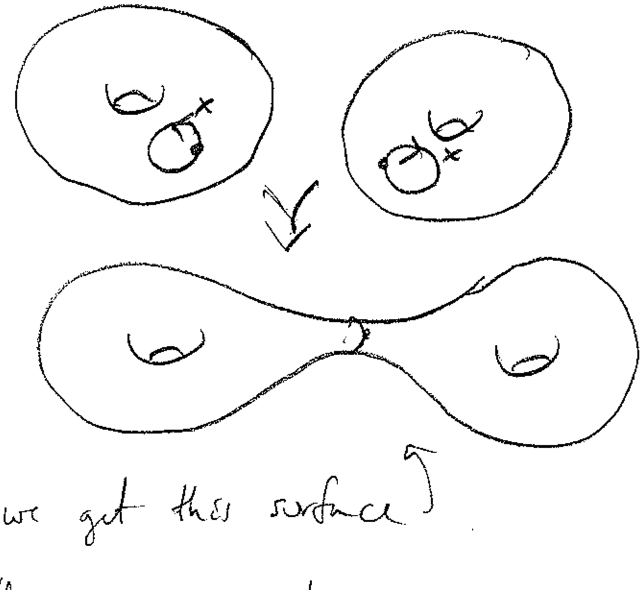


linear homeomorphisms are determined by combinatorial pairing of sides and choice of directions (orientations) on sides.





Same on other side, so get



So, we get this surface

This is the case in general.