Math 428 3/3/11

Defn If X is a top. space, $K \subseteq X$ a subset, then a collection of open sets $\{U_\alpha\}_{\alpha\in J}$ w/ $K \subset \bigcup_{\alpha \in J} U_\alpha$ is called an open cover of K.
K is compact if every open cover has a finite subcover: $\forall$ open covers $\{U_\alpha\}_{\alpha\in J}$, $\exists$ $\alpha_1, \dots, \alpha_n \in J$ st $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a cover of K.
—also refer to a top. space X as compact if $X \subseteq X$ is compact—

Theorem III.15 (Heine-Borel) $K \subset \mathbb{R}^n$ is compact iff K is closed and bounded

proof coming up...

Theorem III.16 $f: X \to Y$ continuous, $K \subset X$ compact, then $f(K)$ is compact.

proof: $\{U_\alpha\}_{\alpha\in J}$ an open cover of $f(K)$, then $\{f^{-1}(U_\alpha)\}_{\alpha \in J}$ is an open cover of K, so let $\alpha_1, \dots, \alpha_k$ be s.t. $K \subset f^{-1}(U_{\alpha_1}) \cup \dots \cup f^{-1}(U_{\alpha_k})$

$\Rightarrow f(K) \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_k}$ $\square$

Combining these we obtain:

Corollary III.17 (Max value theorem) If $\Omega \subset \mathbb{R}^n$ is a closed bounded set, $f: \Omega \to \mathbb{R}$ a continuous function, then f attains its max on Ω: $\exists$ $x \in \Omega$ w/ $f(x) \geq f(y)$ $\forall y \in \Omega$.

proof $f(\Omega)$ is compact hence closed and bounded. $\sup(f(\Omega)) \in f(\Omega)$ or is a limit point of $f(\Omega)$. Since $f(\Omega)$ is closed, $\sup(f(\Omega)) \in f(\Omega)$ $\square$.

Lemma III.18 $[a,b] \subset \mathbb{R}$ is compact for every $a<b \in \mathbb{R}$

proof: Let $\{U_\alpha\}_{\alpha\in J}$ be an open cover. Consider

$$\Omega = \{x \in [a,b] \mid [a,x] \subset U_{\alpha_1}, \ldots \alpha_k \text{ some } \alpha_1, \ldots, \alpha_k \in J\}$$

$a \in \Omega$, so $\Omega \neq \emptyset$.

If $x \in \Omega$, clearly $[a,x] \subset \Omega$. If $b \in \Omega$, $[a,b] \subset \Omega$ and we're done. So, let $x = \sup \Omega$. $x \in B_\varepsilon(x) \subset U_\alpha$ some α. Since $x = \sup \Omega$, $\exists$ $y \in B_\varepsilon(x) \cap \Omega$, and $[a,y] \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$ some k, then $[a, x+\varepsilon/2] \cap [a,b] \subset \Omega$. Therefore $x = b \in \Omega$ and we're done $\square$

Proposition III.19 If $E \subset K \subset X$ is a closed subset of a compact set then E is compact.

proof Exercise III.10 $\square$

> Exercise III.11 Suppose X compact, Y Hausdorff, $f: X \to Y$ continuous injection. Prove f is an embedding

Proposition III.20 If X is Hausdorff, $K \subset X$ compact, then K is closed.

proof: $\forall\, x \in X-K$, $y \in K$ $\exists$ disjoint nbhds $y \in U_{xy}$, $x \in V_{xy}$.

Fix x, then $\{U_{xy}\}_{y\in K}$ is an open cover of K, so $\exists$ $y_1, \ldots, y_k \in K$ s.t. $K \subset U_{xy_1} \cup \cdots \cup U_{xy_k}$. then $V_{xy_1} \cap \cdots \cap V_{xy_k}$ is a nbhd of x disjoint from $U_{xy_1} \cup \cdots \cup U_{xy_k}$, hence also K. so $X \setminus K$ is open & K is closed $\square$

Proposition III.21 If $K_1 \subset X_1, K_2 \subset X_2$ are compact, then $K_1 \times K_2$ is compact.

proof: It suffices to consider open covers $\{U_\alpha \times V_\alpha\}_{\alpha\in J}$ w/ U_α, V_α open in X_1, X_2 respectively. $\forall\, x \in K_1$, $\{x\} \times K_2$ is compact ($\cong K_2$), so $\exists$ $U_{\alpha_1} \times V_{\alpha_1} \cup \cdots \cup U_{\alpha_k} \times V_{\alpha_k} \supset \{x\} \times K_2$, hence $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k} =: U_x \Rightarrow U_x \times V_{\alpha_1} \cup \cdots \cup U_x \times V_{\alpha_k} \supset \{x\} \times K_2$

Then $\{U_x\}_{x\in K_1}$ is an open cover of K_1, have $x_1,\ldots,x_n \in K_1$ w/

$U_{x_1} \cup \ldots \cup U_{x_n} \supset K_1$

so

$$K_1 \times K_2 \subset \left(U_{x_1}\times V_{\alpha_{11}} \cup \ldots \cup U_{x_1}\times V_{\alpha_{1k}}\right) \cup \ldots \cup \left(U_{x_n}\times V_{\alpha_{1n}} \cup \ldots \cup U_{x_n}\times V_{\alpha_{kn}}\right)$$
$$\subset U_{\alpha_{11}}\times V_{\alpha_{11}} \cup \ldots \cup U_{\alpha_{kn}}\times V_{\alpha_{kn}} \quad \square$$

<u>Corollary III.22</u> Any finite product of compact sets is compact.

<u>Proof</u>: Induct $\square$

<u>proof of Heine-Borel</u>: 1st by Lemma III.18 & Cor III.22,

$\forall\ M>0$, $[-M,M]^n \subseteq \mathbb{R}^n$ is compact. So, if $K\subset\mathbb{R}^n$ is closed and bounded, then $K\subset[-M,M]^n$ some $M>0$ and by Prop III.19, K is compact. ✓

Now suppose K is compact. Since $\mathbb{R}^n$ is Hausdorff, K is closed by III.20. If K is not bounded, then $\{B_n(0)\}_{n=1}^{\infty}$ is an open cover of K w/ no finite subcover, this contradiction shows K is bounded $\square$.

<u>Remark</u>: This is not true in an arbitrary metric space — Need $\overline{B_n(x)}$ compact for all $n>0$

Other useful facts about compactness:

<u>Proposition III.23</u> If $K_1 \supseteq K_2 \supseteq \ldots$ are nonempty compact subsets of a Hausdorff space X, then $\bigcap_{i=1}^{\infty} K_i \neq \emptyset$.

False w/out assumptions: $\bigcap_{n=1}^{\infty}(0,1/n)=\emptyset$

$\mathbb{Z}$ w/ $\mathcal{T}=\{\mathbb{Z},\emptyset\}$, then $K_n=\mathbb{Z}\cap[n,\infty)$ is compact & $\bigcap K_n=\emptyset$

proof

$X-K_n=U_n$ is open. If $\bigcap K_i=\emptyset$ then $\{U_n\}_{n=1}^{\infty}$ is an open cover of K_1. Observe that $U_1\subseteq U_2\subseteq\ldots$ so K_1 compact $\Rightarrow\exists N$ s.t.

$U_N\supset K_1\supset K_2\supset\ldots\supset K_N$ $\therefore$ $K_N\subset U_N=X-K_N\Rightarrow K_n=\emptyset$ ↯ ☐

Corollary III.24: If K is compact, Hausdorff, 2^{nd} countable (or 1^{st} countable) Then Any sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\lim_{k\to\infty}x_{n_k}=x\in K$

Defn x a top. space, then $\lim_{n\to\infty}x_n=x$ if $\forall$ nbhds U of x, $\exists N$ s.t. $\forall n\geq N$, $x_n\in U$

proof Let $K_n=\overline{\{x_k\}_{k=n}^{\infty}}\subseteq K$. this is closed, hence compact.
Also $K_1\supseteq K_2\supseteq$. Since K Hausdorff, $\bigcap K_i\neq\emptyset$, let $x\in\bigcap K_i$.

Observe that either $x=x_{n_k}$ for some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ $n_1<n_2<\ldots$ and we're done, or x is a limit point of $\{x_k\}_{k=n}^{\infty}$, $\forall n\geq 1$. So, we assume the latter case. 2^{nd} countability $\Rightarrow\exists$ nbhds $U_1,U_2,U_3,\ldots$ of x st. Any nbhd of x contains some U_i (in fact, this only requires 1^{st} countability). We may assume (taking intersections) that $U_1\supseteq U_2\supseteq U_3\supseteq\ldots$

Now, $\forall j,n$ $U_j\cap\{x_k\}_{k=n}^{\infty}\neq\emptyset$, so let $\{x_{n_j}\}_{j=1}^{\infty}$ be a subsequence (w/ $n_1<n_2<\ldots$) so that $x_{n_j}\in U_j$ $\forall j$. Observe that $x_{n_j}\in U_J$ $\forall j\geq J$.
Then given nbhd U of x, $\exists U_J\subset U$, and $\forall j\geq J$, $x_{n_j}\in U_J\subset U$.

so $\lim_{j\to\infty}x_{n_j}=x$ ☐

[To relate convergent sequences to what we've discussed:]

Exercise III.12 Suppose X is 2^{nd} countable (or 1^{st} countable)

(i) Given $A \subseteq X$, prove $x \in \overline{A}$ iff $\exists$ a sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ w/ $x = \lim_{n\to\infty} x_n$.

(ii) For any space Y, prove $f: X \to Y$ is continuous iff $\lim_{n\to\infty} f(x_n) = f(x)$ whenever $\lim_{n\to\infty} x_n = x$ in X.

We want to move back toward knot theory, consider some more geometric spaces:

Def'n An n-manifold is a second countable, Hausdorff space M such that:

(*) $\forall$ $m \in M$ $\exists$ nbhd U of m and a homeomorphism $f: U \to V \underset{open}{\subseteq} \mathbb{R}^n$.

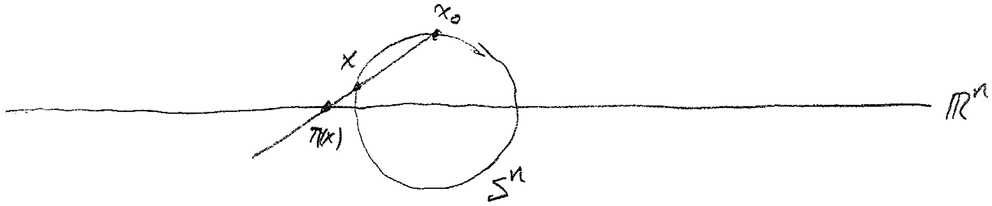
If a space satisfies (*) we say it is locally Euclidean.

Ex • $\mathbb{R}^n$ is an n-manifold (or any n-dim'l vector space)

• If M is an n-manifold, $U \underset{open}{\subseteq} M$, then U is an n-manifold.

• $S^n \subseteq \mathbb{R}^{n+1}$ is an n-manifold: — In fact, for any point $x_0 \in S^n$ $S^n \setminus \{x_0\} \cong \mathbb{R}^n$! (Hausdorff, 2^{nd} countable come from $\subseteq \mathbb{R}^{n+1}$)

To see this, define stereographic projection: $\pi: S^n \setminus \{x_0\} \to \mathbb{R}^n$ by



Exercise III.13 Write down π in terms to coords in $\mathbb{R}^{n+1}$, $\mathbb{R}^n$ for $x_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

and verify it is a homeomorphism

This explains why links in $\mathbb{R}^3$ or S^3 are not so different, and S^3 compact is a little nicer.

• $S^3 \setminus L$ or $\mathbb{R}^3 \setminus L$, L a link, is a 3-manifold.