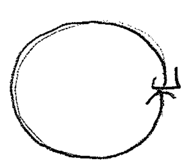
Def A homeomorphism between top. spaces X, Y is a bijection $f: X \longrightarrow Y$ s.t. f and f^{-1} are continuous.

An embedding is a continuous injective map $f: X \longrightarrow Y$ s.t. f is a homeomorphism onto $f(X)$ w/ the subspace topology.

EX $f: [0,1) \longrightarrow \mathbb{C}$, $f(t) = e^{2\pi i t}$ is continuous injection, but not an embedding



$f^{-1}: f([0,1)) = S^1 \longrightarrow [0,1)$ is not continuous $\because (f^{-1})^{-1}([0,\frac{1}{2}))$ is not open in S^1.

Theorem III.7 X, Y, Z top. spaces, then:

(1) $f: X \to Y$, $g: Y \to Z$ continuous, then $g \circ f: X \to Z$ continuous.

(2) If $Y \subseteq X$ is a subset w/ subspace topology, $f: X \to Z$ continuous, then $f|_Y: Y \to Z$ is continuous.

(3) If $X = \bigcup_{\alpha \in J} U_\alpha$, $U_\alpha \subseteq X$ open $\forall \alpha \in J$. $f: X \to Y$ w/ $f|_{U_\alpha}$ continuous $\forall \alpha$, then f is continuous

(4) if $X = A_1 \cup \dots \cup A_k$, $A_i \subseteq X$ closed, $f: X \to Y$, $f|_{A_j}$ continuous $\forall j$ then f is continuous.

(5) $f: X \to Y$ is continuous if $\forall x \in X$ and each nbhd V of $f(x)$, there is nbhd U of x s.t. $f(U) \subseteq V$.

(6) If $\mathcal{B}$ is a basis for Y, then f is continuous iff $f^{-1}(U)$ is open $\forall U \in \mathcal{B}$

proof Exercise III.7 $\square$

↑ ✱ move up to basis defn — A topological space is called 2^{nd} countable if there is a countable basis for the topology

Product topology.

X, Y top. spaces, $X \times Y$ has a natural topology

for which $\pi_X : X \times Y \longrightarrow X$ & $\pi_Y : X \times Y \longrightarrow Y$ are continuous, where:

$\pi_X(x,y) = x \quad \pi_Y(x,y) = y.$

this has as a basis

$\mathcal{B} = \{ U \times V \mid U \subset X \text{ open } V \subset Y \text{ open} \}$ — check this does satisfy properties in Proposition..

[could also just take U, V to be basis elts of X & Y, respectively.]

Generalizes to arbitrary products:

$$\prod_{\alpha \in J} X_\alpha = \left\{ \{x_\alpha\}_{\alpha \in J} \mid x_\alpha \in X_\alpha \right\}$$

by taking a basis

$$\left\{ \prod_{\alpha \in J} U_\alpha \mid U_\alpha \subset X_\alpha \text{ open}, \ U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \right\}$$

This is the "smallest" topology — w.r.t. inclusion as subsets of the power set

s.t. $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$, $\pi_\beta(\{x_\alpha\}_{\alpha \in J}) = x_\beta$, is continuous $\forall \beta$.

Proposition III.8 $f : Y \longrightarrow \prod_{\alpha \in J} X_\alpha$ is continuous iff $f_\beta = \pi_\beta \circ f$ is continuous $\forall \beta$. — observe $f(y) = \{f_\alpha(y)\}_{\alpha \in J}$:

proof: If f is continuous, then since π_β is, so is $\pi_\beta \circ f$ ✓

If $\pi_\beta \circ f$ is continuous, then for any finite set $\beta_1, \dots, \beta_k$, any open $U_{\beta_1}, \dots, U_{\beta_k}$ in $X_{\beta_1}, \dots, X_{\beta_k}$
$f_{\beta_i}^{-1}(U_{\beta_i})$ is open. But $\bigcap_{i=1}^{k} f_{\beta_i}^{-1}(U_{\beta_i}) = f^{-1}\left(\prod_{\beta \in J} U_\beta\right)$ w/ $U_\beta = X_\beta$, $\beta \neq \beta_i$'s

these sets form a basis, so f is continuous. □

Compactness & Connectedness:

Two calculus Theorems: Intermediate value Thrm & Maximum value Theorem — usually skip the proofs. These are naturally Theorems in topology about connectedness & compactness.

Defn X a top. space, a separation of X is a pair of nonempty open sets U & V s.t. ① $U \cap V = \emptyset$
② $U \cup V = X$.

X is connected if there is no separation.

Equivalently, $X \neq \emptyset$ is connected iff the only sets both open & closed are X & $\emptyset$.

Theorem III.9 If $f: X \to Y$ continuous, X connected, then $f(X)$ is connected.

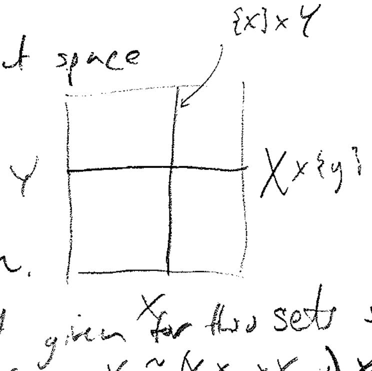
proof if U, V is a separation of $f(X)$, then $f^{-1}(U), f^{-1}(V)$ is a separation of X □

Theorem III.10 If $X_1, \ldots, X_n$ are all connected sets, then $X_1 \times \cdots \times X_n$ is connected.

proof: If $A, B \subset X$ are connected, $A \cap B \neq \emptyset$, then $A \cup B$ is connected: A separation U, V of $A \cup B$ must have one of $U \cap A$ and $V \cap A$ empty [otherwise its a separation of A]. Likewise for B, but then $A \cap B \subset U \cap V = \emptyset$ ↯.

Now look at the product space

$\{x\} \times Y \cup X \times \{y\}$ is connected.



All sets of this type are connected and any two have nonempty intersection.

A similar argument as that given for two sets shows their union, $X \times Y$ is connected.

Now induct on n since $X_1 \times \cdots \times X_n \cong (X_1 \times \cdots \times X_{n-1}) \times X_n$ □

Theorem III.11 The connected subsets of $\mathbb{R}$ are precisely the intervals of any type: $[a,b]$, $[a,b)$, $(a,b]$, (a,b) (allowing $a=-\infty$, $b=\infty$ or $a=b$).

proof If $X \subset \mathbb{R}$ is not an interval, then $\exists\, x \in \mathbb{R} \setminus X$ w/

$X \cap (-\infty, x) \neq \emptyset$ $\quad X \cap (x, \infty) \neq \emptyset$. These sets form a separation of X.

so X is not connected.

Consider now an interval $[a,b]$, $a \neq b$, and suppose $U, V \subset [a,b]$ are disjoint open sets and we prove $U \cup V \neq [a,b]$.

Let $x \in U$, $y \in V$ and wlog assume $x < y$.

Set $U_0 = U \cap [x,y]$, $V_0 = V \cap [x,y]$

Claim $z = \sup U_0 \notin U_0 \cup V_0$

this suffices since z is a limit point of U_0, hence contained in $[x,y] \subset [a,b]$.

proof of claim: If $z \in V_0$, then $(z-\varepsilon, z+\varepsilon) \cap [x,y] \subset V_0$ some $\varepsilon > 0$.

But $(z-\varepsilon, z+\varepsilon) \cap U_0 \neq \emptyset$ $\downarrow\uparrow$ so, $z \notin V_0$.

If $z \in U_0$, $(z-\varepsilon, z+\varepsilon) \cap [x,y] \subset U_0$ some $\varepsilon > 0$. Since $z \neq y \in V_0$,

$z + \varepsilon' \in U_0$ some $\varepsilon' > 0$, but $z = \sup U_0 < z + \varepsilon'$ $\downarrow\uparrow$ so $z \notin U_0$.

This proves the claim, so proves $[a,b]$ connected.

Any interval I is a union of intervals like this $I = \bigcup_{i=1}^{\infty} [a_i, b_i]$ w/ $[a_1, b_1] \subseteq [a_2, b_2] \subseteq \ldots$

since $[a_i, b_i] \cap [a_j, b_j] \neq \emptyset$, I is also connected. $\square$

Corollary III.12 (Intermediate value Theorem)

Any continuous $f: [a,b] \to \mathbb{R}$, any y between $f(a)$ & $f(b)$ is $f(x)$ some $x \in [a,b]$.

proof $f([a,b])$ is connected in $\mathbb{R}$, and therefore is an interval. So it contains all points between any two points in the set. $\square$

Another notion of connectivity is more closely related to geometric intuition:

Defn A path in X is a continuous map $\gamma: [a,b] \to X$. X is path connected if $\forall x,y \in X$ $\exists$ a path $\gamma: [a,b] \to X$ connecting x to y: $\gamma(a) = x$, $\gamma(b) = y$.

Theorem III.13 If X is path connected, it is connected

proof: Exercise III.8 $\square$

Proposition III.14 If $U \subset \mathbb{R}^n$ is open, connected, then it is path connected.

proof Let $x \in U$, consider the sets

$V = \{y \in U \mid \exists \text{ path connecting } x \text{ \& } y\}$ and $U - V$.

Since $B_\varepsilon(y)$ is path connected $\forall y \in \mathbb{R}^n$, $\varepsilon > 0$, we see that V and $U-V$ are open:

since $V \neq \emptyset$ & U is connected, $U - V = \emptyset$ $\square$

[This proof suggests a def'n]

<u>Def'n</u> Given top space X, define two equiv. rel'ns on X:
$x \sim y$ if $\exists$ connected subset $A \subset X$ w/ $x, y \in A$. The equivalence classes are called the (<u>connected</u>) <u>components</u> of X.
Also define <u>path components</u> of X via rel'n $x \approx y$ if $\exists\ A \subset X$ <u>path connected</u>, $x, y \in A$.

<u>Exercise III.9</u> Prove that these are indeed equiv. rel'ns and that any connected (or path connected) subset is contained in a component (or path component, respectively).