Before proceeding, we take a detour into topology — this is the subject which is centered around the notion of continuous functions, and the generality in which this makes sense.

<u>Defn</u> A <u>topology</u> on a set X is a collection of subsets of X, $\mathcal{T} \subset \mathcal{P}(X)$ (= power set of all subsets) satisfying

(1) $\phi, X \in \mathcal{T}$

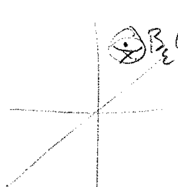
(2) If $\{U_\alpha\}_{\alpha \in J} \subset \mathcal{T}$, then $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$

(3) If $U_1, \ldots, U_k \in \mathcal{T}$ then $(U_1 \cap \cdots \cap U_k) \in \mathcal{T}$.

the sets $U \in \mathcal{T}$ are called <u>open sets</u> (for the topology $\mathcal{T}$)

A set $V \subset X$ is closed if $X - V$ is open.

<u>EX</u> ① $\mathbb{R}^n$, $\mathcal{T} = \{ U \subset \mathbb{R}^n \mid \forall x \in U \ \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subset U \}$



$B_\varepsilon(x) = \{ y \in \mathbb{R}^n \mid \|x - y\| < \varepsilon \}$

— the <u>usual topology</u> on $\mathbb{R}^n$

② If $\mathcal{T}$ is a topology on X, $Y \subset X$, the <u>subspace topology</u> on Y, $\mathcal{T}_Y = \{ U \cap Y \mid U \in \mathcal{T} \}$

③ If X is any set, $\mathcal{T} = \mathcal{P}(X)$ is the <u>discrete topology</u>

(4) If X is any set, $\mathcal{T} = \{U \subset X \mid X - U$ consists of finitely many pts$\} \cup \{\emptyset\}$ is a topology — finite complement topology.

Exercise III.1 How many topologies are there on $\{1,2,3\}$? What are they?

Defn • A topological space is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on X. If $\mathcal{T}$ is understood, we just write X.

• If $(X,\mathcal{T})$ is a top. space, a neighborhood of x is any open set U containing x. [others define a nbhd to be V w/ $x \in U \subseteq V$, U open]

• If $(X,\mathcal{T})$ is a top space, $A \subseteq X$, then $x \in X$ is a limit point of A if $U \setminus \{x\} \cap A \neq \emptyset$ for every nbhd U of x.

Ex In $\mathbb{R}^n$ (w/ usual topology) we have $A \subseteq \mathbb{R}^n$ has $x \in \mathbb{R}^n$ as a limit point if $\forall \varepsilon > 0$, $B_\varepsilon(x) \setminus \{x\} \cap A \neq \emptyset$

$\iff \forall \varepsilon > 0 \; \exists \, x_\varepsilon \in A, x_\varepsilon \neq x$ w/ $\|x - x_\varepsilon\| < \varepsilon$.

Theorem III.1 Given $(X,\mathcal{T})$, $A \subseteq X$ is closed iff A contains all its limit points.

proof If A closed, $X - A$ open which is a nbhd for any $x \in X - A$, so such x is not a limit point, so A contains all its limit points.

If A contains all its limit points, then $\forall x \in X - A$, $\exists$ nbhd U of x w/ $U \cap A = \emptyset$ (since x is not a limit point) so $X \setminus A$ is the union of these nbhds which is itself an open set (by (2) of Defn). $\square$

The collection $\mathcal{C}$ closed subsets clearly satisfies the following "dual" properties to open sets:

(1) $X, \phi \in \mathcal{C}$

(2) if $\{A_\alpha\}_{\alpha\in J} \subset \mathcal{C}$, then $\bigcap_{\alpha\in J} A_\alpha \in \mathcal{C}$.

(3) $A_1, \ldots, A_k \in \mathcal{C} \Rightarrow A_1 \cup \ldots \cup A_k \in \mathcal{C}$.

— can specify a topology in terms of closed sets, too...

<u>Defn</u> The closure of a subset $A \subseteq X$, denoted $\bar{A}$ is the intersection of all closed sets containing A. The <u>interior</u> of A is $A^\circ = X - \cancel{(X - A)}$
$A^\circ =$ union of all open sets contained in A.

<u>Theorem III.2</u> $A \subseteq X$, then $\bar{A} = A \cup \{\text{limit points of } A\}$

<u>Proof</u> Exercise III.2 □

<u>Corollary III.3</u> $A \subseteq X$ is closed iff $A = \bar{A}$.

It is cumbersome to specify <u>all</u> open sets for a topology.

$\mathbb{R}^n$ topology is specified by properties of open sets in terms of $\{B_\varepsilon(x)\}_{x\in\mathbb{R}^n, \varepsilon>0}$...

This is a special case of

<u>Defn</u> If $(X, \mathcal{T})$ is a top. space, then a <u>basis</u> for $\mathcal{T}$ is a collection $\mathcal{B} \subset \mathcal{T}$ s.t. any open set $U \in \mathcal{T}$ is a union of elements of $\mathcal{B}$.

<u>Theorem III.4</u> If $\mathcal{B} \subset \mathcal{P}(X)$ is any set such that

(1) $\bigcup_{B \in \mathcal{B}} B = X$

(2) $\forall\ B_1, \dots, B_k \in \mathcal{B}$, $x \in B_1 \cap \dots \cap B_k$, $\exists\ B_x \in \mathcal{B}$ s.t. $B_x \subset B_1 \cap \dots \cap B_k$.

Then $\exists$ a unique topology for which $\mathcal{B}$ is a basis.

Proof <u>Exercise III.3</u> Check: $\mathcal{T} = \{ \bigcup_{B \in A} B \mid A \subset \mathcal{B} \} \cup \{\emptyset\}$ is a topology $\square$

Another common source of examples:

<u>Defn</u> A <u>metric</u> on a set X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

such that $\forall x, y, z \in X$ we have:

① $d(x,y) \geq 0$ with equality iff $x = y$ (positivity)

② $d(x,y) = d(y,x)$ (symmetric)

③ $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

(X,d) is called a <u>metric space</u>

A metric determines a topology via basis $\mathcal{B} = \{ B_\varepsilon(x) \mid x \in X, \varepsilon > 0 \}$

$B_\varepsilon(x) = \{ y \in X \mid d(x,y) < \varepsilon \}$, called the metric topology

<u>Exercise III.4</u> Check that this does define a topology.

<u>Exercise III.5</u> If $A \subseteq X$, (X,d) a metric space, then $d|_{A \times A}$ is a metric on A and the metric topology on A is the subspace topology from metric topology on X.

In a metric space, the points that are "close" to a point x are those in $B_\varepsilon(x)$. — smaller ε means closer points. A topology does the same thing: points in a nbhd of x are close to x, smaller nbhd means closer pts.

In a metric space, if $x \neq y$, then points close enough to x are not close to y: $d(x,y) = \varepsilon \neq 0$, so

$$B_{\varepsilon/2}(x) \cap B_{\varepsilon/2}(y) = \emptyset.$$

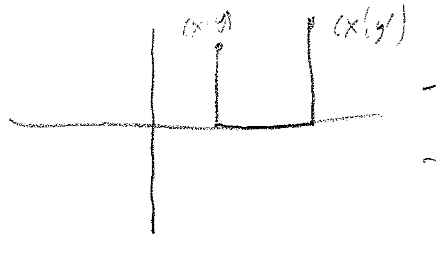
In an arbitrary topological space this need not be true:

Ex Consider the finite complement topology on $\mathbb{R}$. Let x & $y \in \mathbb{R}$ be any two points. Any nbhd of x intersects a nbhd of y.

Defn A topological space X is Hausdorff if for any two points $x \neq y$ in X, $\exists$ nbhds U of x, V of y such that $U \cap V = \emptyset$.

Note: metric topology on any set is always Hausdorff.

Ex Manhattan metric on $\mathbb{R}^2$: $d((x,y),(x',y')) = |x-x'| + |y| + |y'|$



- positive & symmetric ✓
- Δ-ineq. follows from Δ-ineq. in $\mathbb{R}$.

Continuity:

Defn Suppose X, Y are top. spaces. A map $f: X \to Y$ is <u>continuous</u> if for every open $V \subset Y$, $f^{-1}(V)$ is open.

<u>Proposition III.3</u> $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous in this sense iff it is continuous in the ε-δ sense. [Same proof for any metric space]

<u>proof</u>. ($\Rightarrow$) suppose f is continuous as above.

Given $x \in \mathbb{R}^n$, $\varepsilon > 0$, observe that $B_\varepsilon(f(x)) \subset \mathbb{R}^m$ is open. Thus $f^{-1}(B_\varepsilon(f(x)))$ is open and contains $x \in f^{-1}(f(x))$ so, $\exists\ \delta > 0$ s.t. $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$, but then $\forall\ y \in B_\delta(x)$, $f(y) \in B_\varepsilon(f(x))$, i.e. if $\|y - x\| < \delta$, $\|f(x) - f(y)\| < \varepsilon$. ✓

($\Leftarrow$) Exercise III.6 $\square$. — also true for f defined on a subset

<u>Theorem III.6</u> X, Y top. spaces, $f: X \to Y$. Then TFAE.

(1) f is continuous

(2) $\forall\ A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$

(3) $\forall\ B \subset Y$ closed, $f^{-1}(B)$ is closed in X.

<u>proof</u> (1) & (3) are equivalent by basic set theory.

to see (1) $\Rightarrow$ (2), observe that if x is a limit point of A, $\forall$ open V in Y, $f(x) \in V$, $f^{-1}(V) \cap (A \setminus \{x\}) \neq \emptyset$, so $V \cap f(A) \setminus \{f(x)\} \neq \emptyset$ or else $f(x) \in f(A)$

So $f(\bar{A}) \subset \overline{f(A)}$ ✓

(2) $\Rightarrow$ (3): Let $B \subset Y$ be closed, $A = f^{-1}(B)$, then $f(A) \subset B$ and $f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$ so $\bar{A} \subset f^{-1}(B) = A$. Since $A \subset \bar{A}$, we have $A = \bar{A}$ $\square$.