Math 428
Knot Theory
1-18-11

[Introductions
course info.]

I What is knot theory? [see Adams Ch 1].

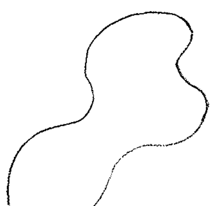
Study of embeddings of one space in another
part of a branch of mathematics called topology,
a close relative of geometry. We will consider just one case.

<u>Defn</u> A <u>knot</u> (or <u>knotted circle</u>) is an embedding
$K : S^1 \longrightarrow \mathbb{R}^3$ (or S^3).

- $\mathbb{R}^k = \{(x_1, \ldots x_k) \mid x_i \in \mathbb{R}\} \supset S^{k-1} = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid \sum_{i=1}^{k} x_i^2 = 1\}$
- embedding = continuous injection.

Often confuse $K : S^1 \longrightarrow \mathbb{R}^3$ w/ image $K = K(S^1) \subset \mathbb{R}^3$ [similarly for S^3].
we call either one a knot.

<u>EX</u>



unknot

trefoil

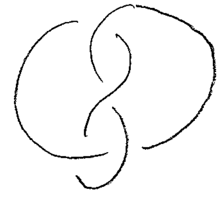


figure eight.

— Yucky —

← Wild knot

We'll be interested in polygonal knots : $K: S^1 \to \mathbb{R}^3$

w/ S^1 decomposed into finitely many arcs, each of which is sent to a straight segment — unless otherwise stated, all knots are assumed polygonal

Pictures? Yes, we will draw many pictures and often our proofs will require the use of pictures.

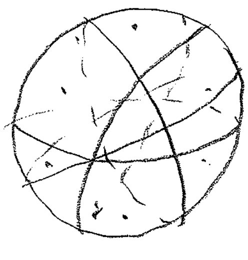
Defn A nice projection of a knot K is the orthogonal projection of K onto a plane $P \subset \mathbb{R}^3$ satisfying:

(1) no edge of K has direction orthogonal to P

(2) the projection is at most 2:1 on K, and is 1:1 "on vertices."
($\pi: \mathbb{R}^3 \to P$ has $\pi^{-1}(\pi(v)) \cap K = \{v\}$ $\forall$ vertices $v \in K$)

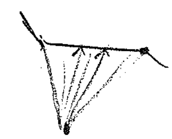
Proposition I.1 Given a knot K, there is a nice projection.

In fact, parametrizing orthogonal projections (up to translation) by $\perp$ unit vectors in S^2 (up to sign), there are "nice" projections corresponding to a subset $X \subset S^2$ which is the complement of a finite # of points and arcs. . .

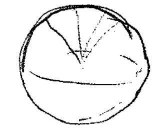


Proof: (1) requires us to omit a finite set of points, namely the unit vectors parallel to the edges of K.

(2) 1:1 on vertices $\Rightarrow$ omit directions from a vertices to edges $\subset$ a finite # of planes



planes define great circles on S^2



2:1 $\Rightarrow$ omit direction of lines going through 3 edges — finitely many curves in S^2 $\square$

Defn A *diagram* D of a knot K is a nice projection of K, together with each *crossing* (point of noninjectivity) indicated as over/under

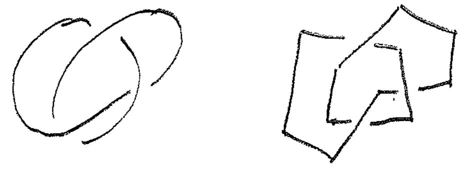
(*) **Equivalence of knots**: Several choices

(1) K & K' are *ambient isotopic* if $\exists$ continuous map

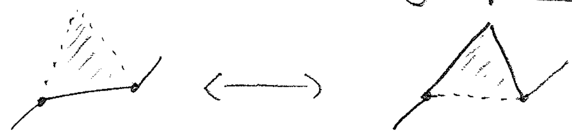
$$H:\mathbb{R}^3\times[0,1]\longrightarrow\mathbb{R}^3 \quad \text{s.t.} \quad H_t(x)=H(x,t) \text{ is a homeomorphism } \forall t\in[0,1]$$

and $H_0=id_{\mathbb{R}^3}$, $H_1(K)=K'$. — $\exists$ a 1-parameter family of homeos deforming K to K'

We often draw diagrams "smoothly" instead of polygonally:



(2) K & K' are <u>combinatorially equivalent</u> if $\exists$ a sequence of moves



the defining triangle  meets knots only along the boundary.

(3) K & K' are <u>homeomorphically equivalent</u> if $\exists$ an orientation preserving homeomorphism taking K to K'.

<u>Theorem I.2</u> All three of these relations are the same for knots.

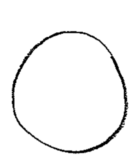
<u>Idea</u> (1) $\Rightarrow$ (3) is clear since $H_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is an o.p. homeo.

(3) $\Rightarrow$ (1) Theorem that any o.p. homeo is isotopic to identity.

(1) $\Leftrightarrow$ (2) (2) is essentially the "discrete" version of (1). We approximate by "piecewise linear" then break up result into small pieces. $\square$

Basic problem: when are two knots equivalent? (we'll just call them "the same") How can we tell?

<u>Ex</u>

 = 

why? - Always go under existing arcs of diagram, so if proj. plane is, say, (x,y)-plane, can assume z-coordinate is decreasing until just before closing up