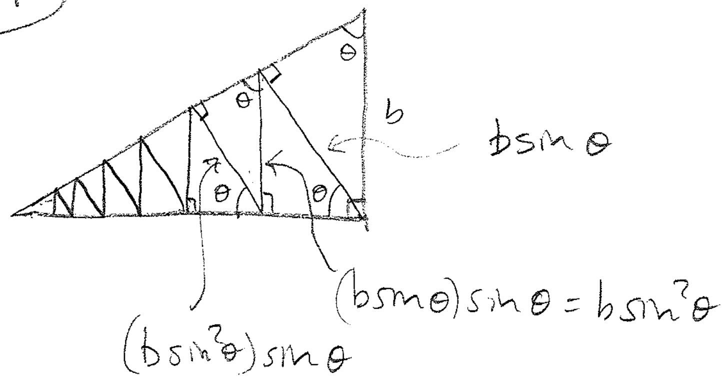
Honors Problem 3

(64)



So, sum of all lengths is

$$\sum_{n=1}^{\infty} b\sin^n\theta = \sum_{n=1}^{\infty}(b\sin\theta)(\sin^{n-1}\theta) = \frac{b\sin\theta}{1-\sin\theta}$$

[this converges since $-1 < \sin\theta < 1$].

(73) ⓐ 1^{st} removed interval has length $= \frac{1}{3}$

all subsequent intervals removed have length $\frac{1}{3}$ · length of previous interval: if l_n = length of n^{th} interval removed, we have

$$l_1 = \frac{1}{3},\ l_2 = \frac{1}{3^2},\ l_3 = \frac{1}{3^3},\ \ldots,\ l_n = \frac{1}{3^n}$$

We are removing 2^{n-1} intervals of length l_n at n^{th} stage, so total length of intervals being removed is

$$l_1 + 2l_2 + 2^2 l_3 + \ldots + 2^{n-1} l_n + \ldots = \sum_{n=1}^{\infty} \frac{1}{3^n} 2^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1/3}{1-2/3} = \frac{1/3}{1/3} = 1$$

since $\left|\frac{2}{3}\right| = \frac{2}{3} < 1$

(b) similar to previous calculation:

A_n = area of n^{th} square removed

$$\left.\begin{aligned} &= \tfrac{1}{9} A_{n-1} \\ \text{and } A_1 &= \tfrac{1}{9} \end{aligned}\right\} \Rightarrow A_n = \left(\tfrac{1}{9}\right)^n$$

we remove 1 square first, then 8 more, then 64 more, then 8^3 more, ..., at n^{th} stage remove 8^{n-1} squares, so total area is

$$\sum_{n=1}^{\infty} 8^{n-1} A_n = \sum_{n=1}^{\infty} \tfrac{1}{9}\left(\tfrac{8}{9}\right)^{n-1} = \underbrace{\frac{1/9}{1-8/9}}_{\left|\frac{8}{9}\right| = 8/9 < 1} = \frac{1/9}{1/9} = 1$$

(76) Let A_n = area of n^{th} circle in the pattern -

we remove 1^{st} circle of area A_1, 3 more of area A_2, 3 more of area A_3, etc. so, total area occupied by circles is

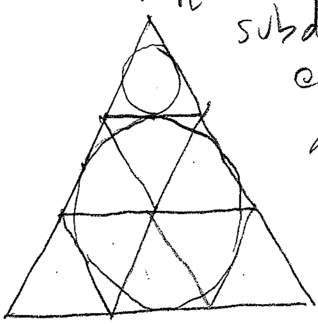
$$A_1 + \sum_{n=2}^{\infty} 3A_n = A_1 + \sum_{n=2}^{\infty} \frac{3}{9^{n-1}} A_1 = A_1\left(1 + \left(\sum_{n=1}^{\infty} 3 \cdot \frac{1}{9^{n-1}}\right) - 3\right)$$

$$= A_1\left(\frac{3}{1-1/9} - 2\right)$$

$$= A_1\left(\frac{27}{8} - \frac{16}{8}\right) = A_1 \frac{11}{8}$$

$$= \frac{\pi 3}{36} \cdot \frac{11}{8} = \boxed{\frac{11\pi}{96}}$$

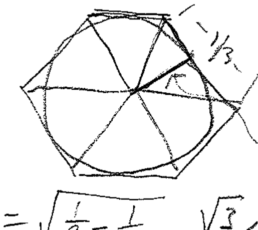
to find A_n: subdivide into 9 equal size equilateral triangles as shown



big circle is inscribed in hexagon in middle

observe that $A_2 = \frac{1}{9} A_1$, $A_3 = \frac{1}{9} A_2 = \frac{1}{9^2} A_1$,

so $A_n = \frac{1}{9^{n-1}} A_1$



$A_1 = \pi r^2$ where

$$r = \sqrt{\tfrac{1}{9} - \tfrac{1}{36}} = \sqrt{3}/6$$