Honors problem 7: complex series.

A *complex series* is a series

$$\sum_{n=0}^{\infty} c_n$$

where each c_n is a complex number (real series are a special case of complex series). We will write $c_n = a_n + ib_n$, where a_n and b_n are the real an imaginary parts of c_n (and hence are real numbers). The real and imaginary parts of the series above are then defined to be

$$\sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} b_n$$

respectively. We say that a complex series converges if and only if *both* the real and imaginary parts converge. In this case, the sum is defined to be the sum of the real part plus i times the sum of the imaginary part:

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n + i \sum_{n=0}^{\infty} b_n$$

1. Prove that if the complex series

$$\sum_{n=0}^{\infty} c_n$$

converges, then

$$\lim_{n\to\infty} |c_n| = 0.$$

(this is the same as saying

$$\lim_{n\to\infty} c_n = 0$$

which means the real and imaginary parts limit to zero.)

Observe that this gives an n^{th} term test for divergence for complex series: if $\lim\limits_{n\to\infty} |c_n| \neq 0$, then the series diverges.

Solution. We write $c_n = a_n + ib_n$, then to say that $\sum c_n$ converges means that $\sum a_n$ and $\sum b_n$ converge. But then $\lim a_n = 0$ and $\lim b_n = 0$ by the n^{th} term test for *real* series. Since $|c_n| = \sqrt{a_n^2 + b_n^2}$, it follows that

$$\lim_{n\to\infty} |c_n| = \lim_{n\to\infty} \sqrt{a_n^2 + b_n^2} = \sqrt{\lim_{n\to\infty} (a_n^2 + b_n^2)} = \sqrt{0+0} = 0.$$

2. The complex series

$$\sum_{n=0}^{\infty} c_n$$

is said to *converge absolutely* if the *real series*

$$\sum_{n=0}^{\infty} |c_n|$$

converges. Prove that if a complex series converges absolutely, then it converges. Hint: observe that $|a_n| \leq |c_n|$ and $|b_n| \leq |c_n|$.

Solution. Since $\sum |c_n|$ converges, and $|a_n| \leq |c_n|$ and $|b_n| \leq |c_n|$, it follows that $\sum |a_n|$ and $\sum |b_n|$ converges (these are *real* series). Therefore, because absolutely convergent real series converge, we see that $\sum a_n$ and $\sum b_n$ both converge. But then by definition, $\sum c_n$ also converges.

From this we see that all the tests for absolute convergence of real series also give tests for absolute convergence of complex series.

A *complex power series centered at a* is an expression of the form

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

where c_n and a are all allowed to be complex numbers and we think of z as a variable. For simplicity, we just consider complex power series centered at 0.
We say that the complex power series

$$\sum_{n=0}^{\infty} c_n z^n$$

converges at $w \in \mathbb{C}$ if the power series converges when the complex number w is substituted in for z:

$$\sum_{n=0}^{\infty} c_n w^n.$$

3. Prove that the complex geometric power series

$$\sum_{n=0}^{\infty} z^n$$

converges absolutely for complex numbers z with $|z| < 1$ and diverges for $|z| \geq 1$. Hint: make sure you use both 1 and 2 above.

Solution. If $|z| < 1$, then

$$\sum_{n=0}^{\infty} |z^n| = \sum_{n=0}^{\infty} |z|^n$$

converges (since $|z|$ is a real number less than 1 and the series is geometric). Therefore, by 2 we know that $\sum z^n$ converges. If $|z| \geq 1$, then $\lim |z^n| = \lim |z|^n$ is not 0, so by 1, the series diverges.

4. Suppose that the power series

$$\sum_{n=0}^{\infty} c_n z^n$$

converges at $w_0 \neq 0$. By part (a) it follows that

$$\lim_{n\to\infty} |c_n w_0^n| = 0.$$

Prove that if $|w| < |w_0|$, then the power series converges absolutely at w. Hint: use the same idea as for real power series.

Solution. This is identical to the proof for real power series. Assuming $\sum c_n w_0^n$ converges, number 1 above implies $\lim |c_n w_0^n| = 0$, and hence there is some constant $C > 0$ so that $|c_n w_0^n| < C$ for all n. Then if $|w| < |w_0|$, we have $|w/w_0| < 1$, and

$$|c_n w_0^n| = \left| c_n w_0^n \frac{w^n}{w_0^n} \right| = |c_n w_0^n| \left| \frac{w}{w_0} \right|^n \leq C \left| \frac{w}{w_0} \right|^n$$

Since $\sum C|w/w_0|^n$ converges by 3, it follows that $\sum |c_n w^n|$ converges, and hence the power series converges absolutely at w.

[[nothing to do here]] From this we obtain the analogue of Theorem 3 from Section 11.8 of the book:

Theorem *Given a power series*

$$\sum_{n=0}^{\infty} c_n z^n$$

exactly one of the following holds

1. *the series converges only for $z = 0$ and diverges for all other $z \in \mathbb{C}$,*
2. *the series converges for every $z \in \mathbb{C}$,*
3. *there is a positive real number $R > 0$ so that the series converges absolutely for $|z| < R$ and diverges for $|z| > R$.*

We can therefore define the *radius of converges* as $0 \leq R \leq \infty$ (with $R = 0$ and $R = \infty$ in cases (1) and (2), respectively) just as for real power series.

5. Prove that the following power series have infinite radius of convergence.

(a.) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$

(b.) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$

(c.) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

We can now *define* e^z, $\sin(z)$ and $\cos(z)$ to be the sums of these series, respectively for every complex number z. From what we have done in class, we know that this agrees with the definitions when z is a *real* number.

Solution. Taking absolute values, this is identical to the proof for the corresponding real power series.

6. Using the previous problem and definitions, prove that for every complex number z we have

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

and conclude that

$$e^{iz} = \cos(z) + i\sin(z).$$

(As a comparison, recall that in the first honors problem, we defined $e^{ix} = \cos(x) + i\sin(x)$ for a real number x. Now we see that this equation is valid not just for real numbers, but for complex numbers, as a consequence of the more natural, power series definitions.)

Solution. We have

$$\begin{aligned}
\frac{e^{iz} + e^{-iz}}{2} &= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} + \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty}\left((i^n + (-i)^n)\frac{z^n}{n!}\right)\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left((1 + (-1)^n)i^n\frac{z^n}{n!}\right)\right) = \frac{1}{2}\left(\sum_{k=0}^{\infty}\left(2i^{2k}\frac{z^{2k}}{(2k)!}\right)\right) \\
&= \sum_{k=0}^{\infty}\left((i^2)^k\frac{z^{2k}}{(2k)!}\right) = \sum_{k=0}^{\infty}\left((-1)^k\frac{z^{2k}}{(2k)!}\right) = \cos(z)
\end{aligned}$$

The computation for $\sin(z)$ is similar.

Finally, just compute

$$\cos(z) + i\sin(z) = \frac{e^{iz} + e^{-iz}}{2} + i\frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz} + e^{iz} - e^{iz}}{2} = e^{iz}.$$

7. We define the hyperbolic cosine and sine just as in the real case

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2} \qquad \text{and} \qquad \sinh(z) = \frac{e^{z} - e^{-z}}{2}.$$

Using these definitions and the previous problem, prove that the following relations hold between the trig functions and the (simpler) hyperbolic trig functions.

$$\cosh(iz) = \cos(z) \qquad \cos(iz) = \cosh(z)$$

and

$$\sinh(iz) = i\sin(z) \qquad \sin(iz) = i\sinh(z).$$

Solution. These are simple substitutions

$$\cosh(iz) = \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$$

$$\cos(iz) = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{z}}{2} = \cosh(z)$$

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i\frac{e^{iz} - e^{-iz}}{2i} = i\sin(z)$$

$$\sin(iz) = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{e^{-z} - e^{z}}{2i} = -\frac{1}{i}\frac{e^{z} - e^{-z}}{2} = i\frac{e^{z} - e^{-z}}{2} = i\sinh(z).$$