A **complex series** is a series

\[ \sum_{n=0}^{\infty} c_n \]

where each \( c_n \) is a complex number (real series are a special case of complex series). We will write \( c_n = a_n + ib_n \), where \( a_n \) and \( b_n \) are the real and imaginary parts of \( c_n \) (and hence are real numbers). The real and imaginary parts of the series above are then defined to be

\[ \sum_{n=0}^{\infty} a_n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n \]

respectively. We say that a complex series converges if and only if both the real and imaginary parts converge. In this case, the sum is defined to be the sum of the real part plus \( i \) times the sum of the imaginary part:

\[ \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n + i \sum_{n=0}^{\infty} b_n \]

1. Prove that if the complex series

\[ \sum_{n=0}^{\infty} c_n \]

cconverges, then

\[ \lim_{n \to \infty} |c_n| = 0. \]

(this is the same as saying

\[ \lim_{n \to \infty} c_n = 0 \]

which means the real and imaginary parts limit to zero.)

Observe that this gives an \( n^{th} \) term test for divergence for complex series: if \( \lim_{n \to \infty} |c_n| \neq 0 \), then the series diverges.

2. The complex series

\[ \sum_{n=0}^{\infty} c_n \]

is said to converge absolutely if the real series

\[ \sum_{n=0}^{\infty} |c_n| \]

converges. Prove that if a complex series converges absolutely, then it converges. Hint: observe that \( |a_n| \leq |c_n| \) and \( |b_n| \leq |c_n| \).

From this we see that all the tests for absolute convergence of real series also give tests for absolute convergence of complex series.
A complex power series centered at \( a \) is an expression of the form

\[
\sum_{n=0}^{\infty} c_n (z - a)^n
\]

where \( c_n \) and \( a \) are all allowed to be complex numbers and we think of \( z \) as a variable. For simplicity, we just consider complex power series centered at 0. We say that the complex power series

\[
\sum_{n=0}^{\infty} c_n z^n
\]

converges at \( w \in \mathbb{C} \) if the power series converges when the complex number \( w \) is substituted in for \( z \):

\[
\sum_{n=0}^{\infty} c_n w^n.
\]

3. Prove that the complex geometric power series

\[
\sum_{n=0}^{\infty} z^n
\]

converges absolutely for complex numbers \( z \) with \( |z| < 1 \) and diverges for \( |z| \geq 1 \). Hint: make sure you use both (a) and (b) above.

4. Suppose that the power series

\[
\sum_{n=0}^{\infty} c_n z^n
\]

converges at \( w_0 \neq 0 \). By part (a) it follows that

\[
\lim_{n \to \infty} |c_n w_0^n| = 0.
\]

Prove that if \( |w| < |w_0| \), then the power series converges absolutely at \( w \). Hint: use the same idea as for real power series.

[[nothing to do here]] From this we obtain the analogue of Theorem 3 from Section 11.8 of the book:

**Theorem** Given a power series

\[
\sum_{n=0}^{\infty} c_n z^n
\]

exactly one of the following holds

1. the series converges only for \( z = 0 \) and diverges for all other \( z \in \mathbb{C} \),
2. the series converges for every \( z \in \mathbb{C} \),
3. there is a positive real number \( R > 0 \) so that the series converges absolutely for \( |z| < R \) and diverges for \( |z| > R \).
We can therefore define the *radius of converges* as $0 \leq R \leq \infty$ (with $R = 0$ and $R = \infty$ in cases (1) and (2), respectively) just as for real power series.

5. Prove that the following power series have infinite radius of convergence.

(a.) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

(b.) $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$

(c.) $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

We can now define $e^z$, $\sin(z)$ and $\cos(z)$ to be the sums of these series, respectively for every complex number $z$. From what we have done in class, we know that this agrees with the definitions when $z$ is a *real* number.

6. Using the previous problem and definitions, prove that for every complex number $z$ we have

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

and conclude that

$$e^{iz} = \cos(z) + i \sin(z).$$

(As a comparison, recall that in the first honors problem, we defined $e^{ix} = \cos(x) + i \sin(x)$ for a real number $x$. Now we see that this equation is valid not just for real numbers, but for complex numbers, as a consequence of the more natural, power series definitions.)

7. We define the hyperbolic cosine and sine just as in the real case

$$\cosh(z) = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh(z) = \frac{e^z - e^{-z}}{2}.$$ 

Using these definitions and the previous problem, prove that the following relations hold between the trig functions and the (simpler) hyperbolic trig functions.

$$\cosh(iz) = \cos(z) \quad \cos(iz) = \cosh(z)$$

and

$$\sinh(iz) = i \sin(z) \quad \sin(iz) = i \sinh(z).$$