Honors problem 4: complex series.

A *complex series* is a series

$$\sum_{n=0}^{\infty} c_n$$

where each c_n is a complex number (real series are a special case of complex series). We will write $c_n = a_n + ib_n$, where a_n and b_n are the real an imaginary parts of c_n (and hence are real numbers). The real and imaginary parts of the series above are then defined to be

$$\sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} b_n$$

respectively. We say that a complex series converges if and only if *both* the real and imaginary parts converge. In this case, the sum is defined to be the sum of the real part plus i times the sum of the imaginary part:

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n + i \sum_{n=0}^{\infty} b_n$$

a) Prove that if the complex series

$$\sum_{n=0}^{\infty} c_n$$

converges, then

$$\lim_{n\to\infty} |c_n| = 0.$$

(this is the same as saying

$$\lim_{n\to\infty} c_n = 0$$

which means the real and imaginary parts limit to zero.)

Observe that this gives an n^{th} term test for divergence for complex series: if $\lim_{n\to\infty} |c_n| \neq 0$, then the series diverges.

b) The complex series

$$\sum_{n=0}^{\infty} c_n$$

is said to *converge absolutely* if the *real series*

$$\sum_{n=0}^{\infty} |c_n|$$

converges. Prove that if a complex series converges absolutely, then it converges. Hint: observe that $|a_n| \leq |c_n|$ and $|b_n| \leq |c_n|$.

From this we see that all the tests for absolute convergence of real series also give tests for absolute convergence of complex series.

A *complex power series centered at a* is an expression of the form

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

where c_n and a are all allowed to be complex numbers and we think of z as a variable. For simplicity, we just consider complex power series centered at 0.
We say that the complex power series

$$\sum_{n=0}^{\infty} c_n z^n$$

converges at $w \in \mathbb{C}$ if the power series converges when the complex number w is substituted in for z:

$$\sum_{n=0}^{\infty} c_n w^n.$$

c) Prove that the complex geometric power series

$$\sum_{n=0}^{\infty} z^n$$

converges absolutely for complex numbers z with $|z| < 1$ and diverges for $|z| \geq 1$. Hint: make sure you use both (a) and (b) above.

d) Suppose that the power series

$$\sum_{n=0}^{\infty} c_n z^n$$

converges at $w_0 \neq 0$. By part (a) it follows that

$$\lim_{n\to\infty} |c_n w_0^n| = 0.$$

Prove that if $|w| < |w_0|$, then the power series converges absolutely at w. Hint: use the same idea as for real power series.

[[nothing to do here]] From this we obtain the analogue of Theorem 3 from Section 11.8 of the book:

Theorem *Given a power series*

$$\sum_{n=0}^{\infty} c_n z^n$$

exactly one of the following holds

1. *the series converges only for* $z = 0$ *and diverges for all other* $z \in \mathbb{C}$,
2. *the series converges for every* $z \in \mathbb{C}$,
3. *there is a positive real number* $R > 0$ *so that the series converges absolutely for* $|z| < R$ *and diverges for* $|z| > R$.

We can therefore talk about the radius of converges as $0 \leq R \leq \infty$ (with $R = 0$ and $R = \infty$ in cases (1) and (2), respectively) just as for real power series.