Math 231 – Honors problem 3 solution

Problem 80, page 686.

Part (a) Given $\epsilon > 0$, the fact that $\lim_{n \to \infty} a_{2n} = L$, we know that there exists some number $N_\epsilon > 0$ so that whenever $2n \geq N_\epsilon$ we have

$$|a_{2n} - L| < \epsilon.$$ 

Similarly, there is some number $N_\delta > 0$ so that whenever $2n + 1 \geq N_\delta$ we have

$$|a_{2n+1} - L| < \epsilon.$$ 

Then, let $N$ be the maximum of the two numbers $N_\epsilon$ and $N_\delta$. If $n \geq N$ and $n$ is even, then $n \geq N_\epsilon$ and

$$|a_n - L| < \epsilon.$$ 

On the other hand, if $n \geq N$ and $n$ is odd, then

$$|a_n - L| < \epsilon.$$ 

Since any natural number $n$ is either even or odd, we see that for $n \geq N$ we have

$$|a_n - L| < \epsilon$$ 

and so

$$\lim_{n \to \infty} a_n = L.$$ 

Part (b) You can easily verify by calculating that for $1 \leq n \leq 3$ we have

$$a_{2n} > a_{2n+2}$$ 

and

$$a_{2n-1} > a_{2n+1}.$$ 

We can prove now prove that the even terms decrease and the odd terms increase by by induction on $n$. The base case is solved by this calculation. Then suppose $a_{2n} > a_{2n+2}$. From this we have

$$\frac{a_{2n}}{1 + a_{2n}} > \frac{a_{2n+2}}{1 + a_{2n+2}}$$

$$\frac{1}{1 + a_{2n}} < \frac{1}{1 + a_{2n+2}}$$

$$\frac{1}{1 + \frac{1}{1 + a_{2n}}} > \frac{1}{1 + \frac{1}{1 + a_{2n+2}}}$$
\[
\begin{align*}
& a_{2n+1} < a_{2n+3} \\
& 1 + a_{2n+1} < 1 + a_{2n+3} \\
& \frac{1}{1 + a_{2n+1}} > \frac{1}{1 + a_{2n+3}} \\
& \frac{1}{1 + a_{2n+1}} > \frac{1}{1 + a_{2n+3}} > \frac{1}{1 + a_{2n+3}}
\end{align*}
\]
as required.

Since \( a_n \geq 0 \) for all \( n \) and

\[
a_n = 1 + \frac{1}{1 + a_{n-1}} \leq 1 + \frac{1}{1} = 2
\]
it follows that the \( \{a_n\} \) is a bounded sequence. By the bounded monotone convergence theorem, the sequence of even and odd terms converge to numbers \( L_e \) and \( L_o \) respectively. Taking a limit of both sides of the equation

\[
a_n = 1 + \frac{1}{1 + a_{n-1}}
\]

for \( n \) even and \( n \) odd, we obtain two equations

\[
L_e = 1 + \frac{1}{1 + L_o} \quad \text{and} \quad L_o = 1 + \frac{1}{1 + L_e}.
\]

Which can be written as

\[
L_e L_o + L_e = L_o = 2 \quad \text{and} \quad L_e L_o = L_e + L_o = 2.
\]

It follows that \( L_e = L_o \) (add the two equations together), if we call this number \( L \) then

\[
L^2 = 2
\]
so \( L = \sqrt{2} \). By part (a), we have \( \lim_{n \to \infty} a_n = \sqrt{2} \).