Honors problem 2: The gamma function.

The **gamma function** is defined by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx, \quad \text{for } z > 0.$$

a) Show that $\Gamma(1) = 1$.

b) Use a comparison test to show that $\int_0^1 x^{z-1} e^{-x}\, dx$ converges for all $z > 0$ (there is nothing to do when $z \geq 1$).

c) Use a comparison test to show that $\int_1^\infty x^{z-1} e^{-x}\, dx$ converges for all $z > 0$. (HINT: any power of x is smaller than any exponential function in x when x is large.) Conclude that the integral defining $\Gamma(z)$ converges for all $z > 0$.

d) Show that $\Gamma(z+1) = z\Gamma(z)$ for all $z > 0$. (HINT: integration by parts.)

e) Use the last part repeatedly to find a beautiful formula for $\Gamma(n+1)$ for all natural numbers n.