Some solutions and notes for honors problem 1

(e) Formally, a proof by induction is what’s required (see below for more on this). Informally, we observe that for \( k = 1 \) this is clear. Then we can just start applying part (d).

\[
z^2 = zz = (|z|(\cos(x) + i\sin(x)))(|z|(\cos(x) + i\sin(x))) = |z|^2(\cos(2x) + i\sin(2x))
\]

\[
z^3 = z^2z = (|z|^2(\cos(2x) + i\sin(2x)))(|z|(\cos(x) + i\sin(x))) = |z|^3(\cos(3x) + i\sin(3x))
\]

\[
z^4 = z^3z = (|z|^3(\cos(3x) + i\sin(3x)))(|z|(\cos(x) + i\sin(x))) = |z|^4(\cos(4x) + i\sin(4x))
\]

It’s clear we can continue to do this for \( z^5, z^6, z^7, \ldots, z^k, \ldots \).

Formally, we assume we can continue to do this up to some integer \( k \), so that

\[
z^k = |z|^k(\cos(kx) + i\sin(kx))
\]

then we verify that it can also be done for the integer \( k + 1 \) appealing to part (d):

\[
z^{k+1} = z^kz = (|z|^k(\cos(kx) + i\sin(kx)))(|z|(\cos(x) + i\sin(x))) = |z|^{k+1}(\cos((k + 1)x) + i\sin((k + 1)x))
\]

This is called a proof by induction, and is a way to prove a statement holds for all natural numbers \( 1, 2, 3, \ldots \). You first prove it for the base case, which is the case \( k = 1 \) (we also proved it for \( k = 2, 3, \) and 4, though that was, strictly speaking, unnecessary). Next you make the inductive assumption, which is that the statement holds for all natural numbers up to some number \( k \). Then, you prove the general case by proving that the statement still holds for the integer \( k + 1 \) under this inductive assumption.

Why is this valid? The point is that you are making rigorous the idea that you can iterate the proof as many times as you want. Since you proved it for the first natural number 1, and you showed that if you could prove it for \( k \), then you could prove it for \( k + 1 \), this means you can prove it for the next natural number 2. By the same reasoning you can prove it for the next natural number 3, and likewise 4, and 5, etc. Since any natural number can be obtained from 1 by adding 1 some number of times, this proves the statement for all natural numbers.

(f) We write \( z = x + iy \) and \( w = u + iv \), then by part (d) and properties of real
exponentials:
\[ e^z e^w = e^{x+iy} e^{u+iv} \]
\[ = e^x e^u (\cos(y) + i \sin(y))(\cos(v) + i \sin(v)) \]
\[ = e^{x+u} (\cos(y + v) + i \sin(y + v)) \]
\[ = e^{x+u} e^{i(y+v)} = e^{x+u+i(y+v)} = e^{z+w} \]

(g) It suffices to prove that the limit of the square is 0. For this we write
\[
\lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right|^2
\]
\[= \lim_{h \to 0} \left| \frac{u(x+h) + iv(x+h) - u(x) - iv(x)}{h} - u'(x) - iv'(x) \right|^2
\]
\[= \lim_{h \to 0} \left( \frac{u(x+h) - u(x)}{h} - u'(x) \right)^2 + i \left( \frac{v(x+h) - v(x)}{h} - v'(x) \right)^2 \]
\[= \lim_{h \to 0} \left( \frac{u(x+h) - u(x)}{h} - u'(x) \right)^2 + \left( \frac{v(x+h) - v(x)}{h} - v'(x) \right)^2 \]

where the last equation follows from your computation in part (a) that for \( z = a + ib \) we have \( |z|^2 = z \overline{z} = a^2 + b^2 \). By definition of the derivative of \( u \) and \( v \) we have
\[ \lim_{h \to 0} \left( \frac{u(x+h) - u(x)}{h} - u'(x) \right)^2 + \lim_{h \to 0} \left( \frac{v(x+h) - v(x)}{h} - v'(x) \right)^2 = 0 + 0 = 0 \]

(h) We have
\[ \frac{d}{dx} (e^{(a+bi)x}) = \frac{d}{dx} (e^{ax}(\cos(bx) + i \sin(bx))) \]
\[= ae^{ax}(\cos(bx) + i \sin(bx)) + e^{ax}(-b \sin(bx) + ib \cos(bx)) \]
\[= e^{ax}(a \cos(bx) - b \sin(bx) + i(a \sin(bx) + b \cos(bx))) \]

On the other hand, we also have
\[ (a+bi)e^{(a+bi)x} = (a+bi)(e^{ax}(\cos(bx) + i \sin(bx))) \]
\[= e^{ax}(a \cos(bx) - b \sin(bx) + i(a \sin(bx) + b \cos(bx))) \]

comparing these two, we obtain the desired result. The integration fact follows from this and the fundamental theorem of calculus for real functions, applied to the real and imaginary parts.
(i) We observe that by part (i) we have
\[
\int e^{(1+i)x} \, dx = \frac{1}{1+i} e^{(1+i)x} + C
\]
so
\[
\int e^x (\cos(x) + i \sin(x)) \, dx = \frac{1-i}{(1+i)(1-i)} (e^x (\cos(x) + i \sin(x)) + C
\]
\[
\int e^x \cos(x) \, dx + i \int e^x \sin(x) \, dx = \frac{1}{2} e^x (\cos(x) - i \cos(x) + i \sin(x) + \sin(x)) + C
\]
\[
\int e^x \cos(x) \, dx + i \int e^x \sin(x) \, dx = \frac{e^x}{2} ((\cos(x) + \sin(x)) + i(\sin(x) - \cos(x))) + C
\]
Because the real and imaginary parts of both sides must be equal we get
\[
\int e^x \cos(x) \, dx = \frac{e^x}{2} (\cos(x) + \sin(x)) + C
\]
and
\[
\int e^x \sin(x) \, dx = \frac{e^x}{2} (\sin(x) - \cos(x)) + C
\]