Honors problem 1: Complex numbers.

Arithmetic of complex numbers

Recall that the complex numbers are formally defined as

$$\mathbb{C} = \{a + bi\}$$

where a and b can be any real numbers and i is treated as a variable (so we can identify the complex numbers with the set of linear polynomials with real coefficients). The numbers a and b are called the **real** and **imaginary parts** of $a + bi$, respectively. We view the real numbers as a subset of the complex numbers by identifying the real number a with $a + 0i$. The **imaginary numbers** are the numbers of the form $0 + bi$ for any nonzero real number b.

Addition in the complex numbers is the same as in the set of linear polynomials and is given by

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

Multiplication is defined by

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

and can be thought of as multiplication of polynomials, subject to the condition that $i^2 = -1$. Indeed, notice that if we consider the imaginary number $i = 0 + i$, then we have $i^2 = -1 = -1 + 0i$. Also observe that this is clearly commutative.

The complex number $z = a + bi$ is said to have **real part** a and **imaginary part** b. We define the **conjugate** of z by

$$\overline{z} = a - bi.$$

(a.) Check that $z\overline{z}$ is a positive real number.

(b.) Check that the real and imaginary parts of z are given by

$$\frac{z + \overline{z}}{2} \quad \text{and} \quad \frac{i(\overline{z} - z)}{2},$$

respectively.

According to part (a.), we can define the **absolute value** of a complex number z to be the positive square root

$$|z| = \sqrt{z\overline{z}}.$$

We write the multiplicative inverse of a nonzero complex number z as $1/z$, just as in real case (so, $z \cdot 1/z = 1 = 1 + 0i$). Note that $1/z = (1/|z|^2) \cdot \overline{z}$. Division z/w is just multiplication by the multiplicative inverse $z/w = z \cdot 1/w$.

Geometry of complex numbers

We can visualize the complex numbers by identifying the complex number $a + bi$ with a point (a, b) in the plane $\mathbb{R}^2$. Then observe that $|z|$ is simply the distance to the origin in the plane.

(c.) Let z be a complex number. Verify that there is a real number θ so that

$$z = |z|(\cos(\theta) + i\sin(\theta)).$$

(d.) Verify that if z is not zero, then the number θ from part (b) is unique up to integral multiples of 2π. That is, if we write

$$z = |z|(\cos(\theta) + i\sin(\theta)) = |z|(\cos(\theta') + i\sin(\theta'))$$

for some real numbers θ and θ', then $\theta - \theta' = 2k\pi$ for some integer k.

(e.) If $z = |z|(\cos(\theta) + i\sin(\theta))$ and $w = |w|(\cos(\psi) + i\sin(\psi))$, then verify that

$$z \cdot w = |z||w|(\cos(\theta + \psi) + i\sin(\theta + \psi)).$$

(f.) If $z = |z|(\cos(\theta) + i\sin(\theta))$, verify that for any non-negative integer k, we have

$$z^k = |z|^k(\cos(k\theta) + i\sin(k\theta)).$$

(g.) Writing $e^{x+iy} = e^x(\cos(y) + i\sin(y))$ verify that

$$e^z e^w = e^{z+w}.$$

Recall that if $f : \mathbb{R} \to \mathbb{C}$ is a function, then there are real valued functions $u(x)$ and $v(x)$ so that $f(x) = u(x) + iv(x)$. Writing f in this way we have defined

$$f'(x) = u'(x) + iv'(x) \text{ and } \int f(x)dx = \int u(x)dx + i\int v(x)dx$$

when u and v are differentiable (at x) and integrable, respectively. In this case we say that f is differentiable (at x) and integrable, respectively.

(h.) Show that with this definition, if f is differentiable at x then

$$\lim_{h\to 0}\left|\frac{f(x+h) - f(x)}{h} - f'(x)\right| = 0.$$

(i.) Verify that for any $a + bi$ we have

$$\frac{d}{dx}e^{(a+bi)x} = (a + bi)e^{(a+bi)x}$$

and consequently we have

$$\int e^{(a+bi)x}dx = \frac{1}{a+bi}e^{(a+bi)x} + C.$$

(j.) Using integration by parts we were able to compute the integrals

$$\int e^x\cos(x)dx \text{ and } \int e^x\sin(x)dx.$$

Compute these integrals using (g) and the definition of integrals of complex valued functions *instead of integration by parts.*