Differential Geometry: Problem set 7

November 14, 2006

Due Wednesday, November 29

1. Consider the following vector field and 1–form on $\mathbb{R}^3$

$$\xi = (x^2 + x^3)\frac{\partial}{\partial x^1} + (x^1 + x^3)\frac{\partial}{\partial x^2} + (x^1 + x^2)\frac{\partial}{\partial x^3} \in \mathfrak{X}(\mathbb{R}^3)$$

$$\alpha = (x^1)^2 dx^1 + (x^2)^2 dx^2 + (x^3)^2 dx^3 \in \Omega^1(\mathbb{R}^3)$$

Compute

$$L_\xi \alpha$$

2. The 1–ball $\mathbb{B}_1^n \subset \mathbb{R}^n$ is a smooth manifold with boundary. Check that the top form on S^{n-1} given by problem 6 of homework 6 defines the boundary orientation of $S^{n-1} = \partial \mathbb{B}_1^n$.

   Compute

$$\int_{S^{n-1}} \omega$$

   [Hint: reduce it to an integral you know or can find somewhere....]

3. Let $f : M \to N$ be a smooth map. Check that

$$f^* : \Omega^* N \to \Omega^* M$$

   descends to a homomorphism

$$f^* : H^*_{dR}(N) \to H^*_{dR}(M)$$

4. (a) Check that

$$\wedge : H^p_{dR}(M) \times H^{n-p}_{dR}(M) \to H^n_{dR}(M)$$

   given by

$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$$

   is well-defined and bilinear.

(b) If M is closed (i.e. compact without boundary) and orientable prove that

$$\dim(H^n_{dR}(M)) \geq 1$$

(c) If M is closed oriented, then check that

$$\int \wedge : H^p_{dR}(M) \times H^{n-p}_{dR}(M) \to \mathbb{R}$$

given by

$$[\alpha], [\beta] \mapsto \int_M \alpha \wedge \beta$$

is a well-defined bilinear map.

5. Give S^n the induced Riemannian metric from the (standard) embedding into $\mathbb{R}^{n+1}$ and compute in stereographic coordinates from the north pole $(0, 0, ..., 0, 1)$ the metric g_{ij}.

6. One model of **hyperbolic n–space** is the Riemannian n–manifold

$$\mathbb{H}^n = \{(x^1, ..., x^n) \,|\, x^n > 0\}$$

with the Riemannian metric

$$g_{ij} = \frac{\delta_{ij}}{(x^n)^2}$$

That is, the Euclidean inner product scaled by the reciprocal of the height squared.

(a) Given $(t^1, ..., t^{n-1}) \in \mathbb{R}^{n-1}$, check that the map

$$f(x^1, ..., x^n) = (x^1 + t^1, ..., x^{n-1} + t^{n-1}, x^n)$$

is an isometry.

(b) Given $\lambda \in \mathbb{R}^+$ (the positive real numbers) check that

$$g(x^1, ..., x^n) = (\lambda x^1, ..., \lambda x^n)$$

is an isometry.