

Differential Geometry: Problem set 6

November 17, 2006

Due Friday, November 13

1. Let V be a finite dimensional vector space. Verify that $v_1 \wedge \dots \wedge v_k \in \Lambda_k V$ is nonzero if and only if v_1, \dots, v_k are linearly independent.
2. In class we defined a canonical isomorphism indirectly

$$\Lambda^k V \cong \text{Alt}_k(V; \mathbb{R})$$

where $\text{Alt}_k(V; \mathbb{R})$ was the space of alternating maps from the k -fold product of V with itself to \mathbb{R} . However, the isomorphism was somewhat indirect. To check your understanding, given

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1, \dots, i_k} e^{i_1} \wedge \dots \wedge e^{i_k} \in \Lambda^k V$$

write down the effect of the associated alternating function on the k -tuple $(e_{j_1}, \dots, e_{j_k})$.

3. Let U, V and W be finite dimensional vector spaces and $T : U \rightarrow V$ and $S : V \rightarrow W$ are linear maps. Check that the induced maps

$$T^* : \Lambda^* V \rightarrow \Lambda^* U \quad S^* : \Lambda^* W \rightarrow \Lambda^* V \quad (ST)^* : \Lambda^* W \rightarrow \Lambda^* U$$

satisfy

$$(ST)^* = T^* S^*$$

4. Suppose M, N and L are smooth manifolds with smooth maps

$$F : M \rightarrow N \quad G : N \rightarrow L$$

check that the induced maps

$$F^* : \Omega^* N \rightarrow \Omega^* M \quad G^* : \Omega^* L \rightarrow \Omega^* N \quad (GF)^* : \Omega^* L \rightarrow \Omega^* M$$

satisfy

$$(GF)^* = F^* G^*$$

5. Verify that if $F : M \rightarrow N$ is a smooth map and $\omega \in \Omega^p(N)$, then

$$F^*(d\omega) = dF^*(\omega)$$

6. In \mathbb{R}^n , define an $n - 1$ -form

$$\omega = \sum_{i=1}^n (-1)^{i+1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

(where as usual $\widehat{\cdot}$ means omit \cdot). Consider the standard embedding

$$f : S^{n-1} \subset \mathbb{R}^n$$

and prove that $f^*(\omega)$ is a nowhere vanishing top-form, and so S^n is orientable.

7. If n is odd, show that $\mathbb{R}P^n$ is orientable. Show that when n is even it is non-orientable. In both cases, it is useful to consider the quotient map $S^n \rightarrow \mathbb{R}P^n$.
8. Let M be any smooth manifold and $\pi : T^*M \rightarrow M$ its cotangent bundle. There is a canonically defined 1-form $\Theta \in \Omega^1(T^*M)$ defined by

$$\Theta_{m,\alpha} = (d\pi_{(m,\alpha)})^* \alpha$$

That is, at the point (m, α) , $\Theta_{(m,\alpha)} \in T_{(m,\alpha)}^* T^*M$ is the pull-back by $d\pi^*$ of the covector $\alpha \in T_m^*(M)$. Verify in local coordinates that Θ is indeed a (smooth) 1-form. Specifically, if x^1, \dots, x^n are coordinates on M and $x^1, \dots, x^n, y^1, \dots, y^n$ are the associated coordinates on T^*M , express Θ in terms of these coordinates and verify it is smooth. Also compute (in these coordinates) $\omega = d\Theta$ and verify that the n -fold wedge product

$$\omega \wedge \dots \wedge \omega$$

is a nowhere vanishing top form.