Differential Geometry: Problem set 5

October 18, 2006

Due Wednesday October 25

1. Given $\xi \in \mathfrak{X}(M)$ with local flow ϕ and $f \in C^\infty(M)$, one can define $L_\xi(f) \in C^\infty(M)$ by
$$(L_\xi(f))(m) = \frac{d}{dt}|_{t=0}(\phi_t^*(f))(m)$$
where (as usual) $\phi_t^*(f) = f \circ \phi_t$. Verify that $L_\xi(f) = \xi(f)$.

2. Consider the vector fields ξ and η on $\mathbb{R}^4$ defined by
$$\xi(x^1, x^2, x^3, x^4) = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^4}$$
and
$$\eta(x^1, x^2, x^3, x^4) = x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^4}$$
   a. Prove that $\phi_t(\mathbf{x}) = e^t \mathbf{x}$ is a complete flow for ξ.
   b. Compute $[\xi, \eta]$ and verify directly that it is equal to $L_\xi \eta$ (so, use the flow ϕ to compute $L_\xi \eta$).

3. For $k \geq 2$ we choose two different identifications $\mathbb{R}^{2k} \cong \mathbb{C}^k$ by
$$F(x^1, ..., x^{2k}) = (x^1 + ix^2, ..., x^{2k-1} + ix^{2k})$$
$$G(x^1, ..., x^{2k}) = (x^2 + ix^3, ..., x^{2k-2} + ix^{2k-1}, x^{2k} + ix^1)$$
There is an action of $\mathbb{R}$ on $\mathbb{C}^k$ given by
$$\mu_t(\mathbf{z}) = e^{it}\mathbf{z}$$
This pulls back via F and G to two actions of $\mathbb{R}$ on $\mathbb{R}^{2k}$.
$$\phi : \mathbb{R} \times \mathbb{R}^{2k} \to \mathbb{R}^{2k}$$
$$\psi : \mathbb{R} \times \mathbb{R}^{2k} \to \mathbb{R}^{2k}$$
respectively.

a. Find vector fields ξ and η that generate the flows ϕ and ψ, respectively.

b. Compute $[\xi, \eta]$ and verify directly that it is equal to $L_\xi \eta$ (using the flow ϕ).

c. Show that ϕ and η restrict to flows on S^{2k-1} (compare with homework 3... not much to do here).

4. [1] Given $A \in \mathrm{M}_{n\times n}(\mathbb{R})$, define

$$\exp(A) = e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

a. Check that

$$\exp : \mathrm{M}_{n\times n}(\mathbb{R}) \to \mathrm{M}_{n\times n}(\mathbb{R})$$

is well defined, and in fact smooth. Hint: consider coordinates $\{y^i_j\}_{i,j=1,\ldots,n}$ on $\mathrm{M}_{n\times n}(\mathbb{R})$, and verify that the coordinate functions of exp are convergent power series.

b. Check that if A and B commute (i.e. $AB = BA$), then

$$\exp(A + B) = \exp(A)\exp(B)$$

c. Verify that exp maps $\mathrm{M}_{n\times n}(\mathbb{R})$ to $\mathrm{GL}_n(\mathbb{R})$. Moreover, viewing a 1–dimensional subspace of $\mathrm{M}_{n\times n}(\mathbb{R})$ as an abelian group, check that the restriction of exp to such a group defines a homomorphism to $\mathrm{GL}_n(\mathbb{R})$.

d. Recall from homework 4 the construction of the vector field ξ^A from $A \in \mathrm{M}_{n\times n}(\mathbb{R})$. Verify that the flow defined by ξ^A

$$\phi^A : \mathbb{R} \times \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$$

is given by

$$\phi^A_t(X) = X \exp(tA)$$

Just observe (nothing to check here) that the action of $\mathbb{R}$ defined by ϕ^A is actually the restriction to the subgroup of part (c.) of the right action of $\mathrm{GL}_n(\mathbb{R})$ on itself.

5. Let V be a finite dimensional vector space. Show that the map

$$\mathbb{R} \otimes V \to V$$

given by

$$t \otimes v \to tv$$

defines an isomorphism $\mathbb{R} \otimes V \cong V$.

[1] if you need to skip a problem, skip this one

6. If $\{v_i\}$ is a (finite) basis for V with dual basis $\{v^i\}$ and $\{u_j\}$ a (finite) basis for U, then verify that the isomorphism $V^* \otimes U \to Hom(V, U)$ takes the basis $v^i \otimes u_j$ to the homomorphism $v^i(\cdot)u_j$, i.e. the homomorphism defined by $v \mapsto v^i(v)u_j$. Verify that with respect to the basis $\{v_i\}$ and $\{u_j\}$ that this homomorphism is described by the matrix E^i_j: the matrix with all zeros except in the $\binom{i}{j}$ entry which is a 1.