Differential Geometry: Problem set 4

September 21, 2006

Due Wednesday October 4

GP problems §1.6, problems 1,9.

1. Let $V, W \subset \mathbb{R}^n$ be subspaces.

   a. Check that $V \pitchfork W$ if and only if $\dim(V) + \dim(W) - \dim(V \cap W) = n$.

   b. Let $\pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^{n-1}$ be the usual quotient map. Check that if $V \pitchfork W$, then $\pi(V \setminus \{0\}) \pitchfork \pi(W \setminus \{0\})$. Is the converse true?

2. Let M be a smooth manifold, show that the map

   $$\sigma : M \to TM$$

   given by

   $$\sigma(m) = (m, 0)$$

   is an embedding. This is called the **zero section** of $\pi : TM \to M$. More generally, any smooth map $s : M \to TM$, for which $\pi \circ s$, is called a **section** of M. A section of $\pi : TM \to M$ is also called a vector field.

3. Suppose $f : M \to N$ is a smooth map.

   a. Show that the map

   $$df : TM \to TN$$

   given by

   $$df(m, \xi) = (f(m), df_m(\xi))$$

   is smooth.

   b. If f is a submersion (respectively, immersion or embedding), then df is also a submersion (respectively, immersion or embedding).

4. If $M \subset N \subset \mathbb{R}^K$ are submanifolds, let

$$TN|_M = \{(m,\xi)\,|\, m \in M\,,\, \xi \in T_m N\}$$

denote the restriction of the tangent bundle of N to M and

$$\mathcal{N}(M) = \{(m,\xi)\,|\, m \in M, \xi \in T_m N, \xi \perp T_m M\}$$

denote the **normal bundle** of M in N.

   a. Prove that $\mathcal{N}(M) \subset TN|_M \subset TN$ are submanifolds.
   b. Prove that the (obvious) projections $\pi : \mathcal{N}(M) \to M$ and $\mu : TN|_M \to M$ are submersions.
   c. Prove that $\sigma : M \to \mathcal{N}(M)$, given by $\sigma(m) = (m, 0)$ is an embedding. This is called the zero section of $\pi : \mathcal{N}(M) \to M$.

5. Let $M \subset \mathbb{R}^n$ be a closed submanifold and $\mathcal{N}(M)$ the normal bundle of M in $\mathbb{R}^n$. Show that the map

$$F : \mathcal{N}(M) \to \mathbb{R}^n$$

given by

$$F(m,\xi) = m + \xi$$

is a diffeomorphism when restricted to some neighborhood U of $\sigma(M) \subset \mathcal{N}(\mathcal{M})$.
(If you get stuck, look in GP.)

6. Let U be an open set in Euclidean space. Check that

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right] = 0$$

7. Let $U \subset \mathbb{R}^3$ and

$$\xi = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - zx\frac{\partial}{\partial z} \text{ and } \eta = y^2\frac{\partial}{\partial x} - xyz\frac{\partial}{\partial z}$$

compute $[\xi, \eta]$.

8. Let M be a smooth manifold and $\xi, \eta, \mu \in \mathfrak{X}(M)$, smooth vector fields on M. Verify the **Jacobi identity**:

$$[[\xi,\eta],\zeta] + [[\eta,\zeta],\xi] + [[\zeta,\xi],\eta] = 0$$

Along with the property $[\xi,\eta] = -[\eta,\xi]$, this shows that the (real vector) space $\mathfrak{X}(M)$ with the bilinear operation $[\cdot,\cdot]$ is a **Lie algebra**.

9. Recall that we have a canonical identification $T_X(\mathrm{GL}_n(\mathbb{R})) = \mathrm{M}_{n\times n}(\mathbb{R})$ for any $X \in \mathrm{GL}_n(\mathbb{R})$. Given $A \in \mathrm{M}_{n\times n}(\mathbb{R})$, we can construct a smooth vector field on $\mathrm{GL}_n(\mathbb{R})$

$$\xi^A : \mathrm{GL}_n(\mathbb{R}) \to T\mathrm{GL}_n(\mathbb{R})$$

by

$$\xi^A(X) = (X, XA)$$

Note that left multiplication by X

$$\ell_X : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$$

$$\ell_X(Y) = XY$$

is actually the restriction to $\mathrm{GL}_n(\mathbb{R})$ of a linear map $\mathrm{M}_{n\times n}(\mathbb{R}) \to \mathrm{M}_{n\times n}(\mathbb{R})$. So the derivative map

$$d(\ell_X)_Y : \mathrm{M}_{n\times n}(\mathbb{R}) \to \mathrm{M}_{n\times n}(\mathbb{R})$$

is just given by

$$d(\ell_X)_Y(B) = XB$$

for any $Y \in \mathrm{GL}_n(\mathbb{R})$.

a. Check that

$$\beta : \mathrm{M}_{n\times n}(\mathbb{R}) \to \mathfrak{X}(\mathrm{GL}_n(\mathbb{R}))$$

given by

$$\beta(A) = \xi^A$$

is a linear injection. Thus we have a finite dimensional subspace of $\mathfrak{X}(\mathrm{GL}_n(\mathbb{R}))$. The image is denoted $\mathfrak{gl}_n(\mathbb{R})$.

b. Check that for every $X \in \mathrm{GL}_n(\mathbb{R})$ the diffeomorphism ℓ_X preserves ξ^A (that is, ξ^A is ℓ_X–related to itself).

c. Check that $\mathfrak{gl}_n(\mathbb{R})$ is closed under the Lie bracket, and in fact:

$$[\xi^A, \xi^B] = \xi^{AB-BA}$$

This last matrix $AB - BA$ is often denoted $[A, B] = AB - BA$, so that the equation becomes

$$[\xi^A, \xi^B] = \xi^{[A,B]}$$

Thus, $\mathfrak{gl}_n(\mathbb{R})$ is a finite dimensional Lie subalgebra of $\mathfrak{X}(\mathrm{GL}_n(\mathbb{R}))$ and is called **the Lie algebra** of $\mathrm{GL}_n(\mathbb{R})$.

d. Check that if A and B are in $T_I(\mathrm{SL}_n(\mathbb{R}))$, i.e. they have trace zero, then $[A, B] \in T_I(\mathrm{SL}_n(\mathbb{R}))$. In particular, after restricting both domain and range, β describes a linear injection of $T_I(\mathrm{SL}_n(\mathbb{R}))$ into $\mathfrak{X}(\mathrm{SL}_n(\mathbb{R}))$ and the image is also closed under Lie bracket. This image $\mathfrak{sl}_n(\mathbb{R})$ is the Lie algebra of $\mathrm{SL}_n(\mathbb{R})$.

e. Similarly, for the orthogonal group $\mathrm{O}_n(\mathbb{R})$, check that if A and B are in $T_I(\mathrm{O}_n(\mathbb{R}))$, i.e. $A + A^t = 0$ and $B + B^t = 0$ (see GP problem 10, §1.5), then $[A, B] \in T_I(\mathrm{O}(\mathbb{R}))$. Again, after restricting appropriately, β defines a linear injection of $T_I(\mathrm{O}(\mathbb{R}))$ into $\mathfrak{X}(\mathrm{O}_n(\mathbb{R}))$. The image $\mathfrak{o}_n(\mathbb{R})$ is the Lie algebra of $\mathrm{O}_n(\mathbb{R})$.

Note that the Lie algebras $\mathfrak{gl}_n(\mathbb{R})$ is isomorphic to $M_{n\times n}(\mathbb{R})$ with the bracket given by $[A, B] = AB - BA$. Similiarly, $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{o}_n(\mathbb{R})$ are isomorphic to Lie subalgebras of $M_{n\times n}(\mathbb{R})$.