Differential Geometry: Problem set 2

August 30, 2006

Due Friday September 8

Read GP §1.1 and §1.2 again. Do problems 8, 9(a)(b), and 11 from §1.2 (use the GP definition of tangent vector).
Read Boothby II.1–II.4 and III.1–III.3.

1. If $M \subset \mathbb{R}^N$ is a smooth manifold in the sense of GP, then it is also a smooth manifold in our sense, as you proved on the last homework. Prove that the map $[\gamma] \to \gamma'(0) \in \mathbb{R}^N$ defines an isomorphism from the tangent space $T_m M$ in our sense to the tangent space in the sense of GP. Again, it might be convenient to define something like $V_m M \subset \mathbb{R}^N$ to be the GP tangent space to M at m, leaving $T_m M$ to denote our tangent space.

2. This problem defines and gives practice with the index notation. Note carefully the placement of superscripts and subscripts in what follows; the name of the index doesn't matter, but the placement does.

   Let V be an n–dimensional vector space. Suppose $\{e_j\}$ and $\{f_j\}$ are two bases for V related by the equation

   $$\sum_i P^i_j f_i = e_j$$

   where $P = (P^i_j)$ is an invertible matrix. Here i is the row index and j is the column index.

   (a.) Suppose that $\xi \in V$ is a vector. Then we can find real numbers ξ^j and $\tilde{\xi}^i$ so that

   $$\xi = \sum_j \xi^j e_j = \sum_i \tilde{\xi}^i f_i$$

   Express $\tilde{\xi}^i$ in terms of ξ^j.

(b.) Suppose $T : V \to V$ is a linear transformation. Relative to the basis $\{e_j\}$ it is expressed as the matrix A defined by

$$Te_j = \sum_i A^i_j e_i$$

and relative to the basis $\{f_i\}$ it is expressed as the matrix B defined by

$$Tf_i = \sum_j B^j_i f_j$$

What is the relationship between A and B?

(c.) The dual space V^* is the vector space of all linear functionals $V \to \mathbb{R}$; it is also n–dimensional. Every basis of V gives rise to a dual basis of V^*. For the basis $\{e_i\}$ of V, the dual basis $\{e^i\}$ of V^* is defined by the equation

$$e^i(e_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

(This equation defines the symbol δ^i_j.) The dual basis $\{f^j\}$ is defined similarly. Express f^j in terms of e^i.

(d.) Suppose $\omega \in V^*$. Then we define its components relative to the basis $\{e^i\}$ by the equation $\omega = \sum_i \omega_i e^i$ and its components relative to the basis $\{f^j\}$ by the equation $\omega = \sum_j \tilde{\omega}_j f^j$. Express the ω_i in terms of $\tilde{\omega}_j$.

(e.) Now suppose $\{h_i\}$ is a basis for the m–dimensional vector space W, and that $T : V \to W$ is a linear transformation. Relative to the bases $\{e_i\}$ and $\{h_j\}$ the map T is expressed as the matrix A defined by

$$Te_j = \sum_i A^i_j h_i$$

The matrix B for the induced map

$$T^* : W^* \to V^*$$

relative to the dual bases $\{e^j\}$ and $\{h^j\}$ is defined by

$$Th^j = \sum_i B^j_i e^i$$

What's the relationship between A and B?

3. Suppose G acts properly discontinuously and freely on a locally compact Hausdorff topological space X.

   (a.) Prove that X/G is Hausdorff.

   (b.) Prove that the quotient map

   $$\pi : X \to X/G$$

   is a **covering map**: for every $x \in X/G$, there is a neighborhood U of x so that the preimage of U is a disjoint union of open sets

   $$\pi^{-1}(U) = \bigsqcup_{\alpha \in J} U_\alpha$$

   with the property that $\pi|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism for every $\alpha \in J$.

   (c.) Prove that if X has a countable basis for the topology, then so does X/G.

   Hints: Recall that a locally compact Hausdorff space has the property that in any neighborhood U about a point x, there is smaller neighborhood V of x with $\overline{V} \subset U$ and $\overline{V}$ compact. Don't be afraid to look at Boothby III.2.

4. Suppose that M is a smooth manifold with a properly discontinuous free smooth action of G on M. Define a smooth structure on M for which

   $$\pi : M \to M/G$$

   is smooth.

5. Suppose that $\pi : M \to N$ is a covering map of topological spaces (see above), and that N is a smooth manifold. If M has countable fibers (meaning $\pi^{-1}(p)$ is countable for every $p \in N$) then construct a smooth structure on M making π into a smooth map.

6. Verify that the following actions are smooth, properly discontinuous, and free (so the quotients are smooth manifolds).

   (a.) $\mathbb{Z}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $A \cdot B = A + B$. The orbit space $\mathbb{R}^n/\mathbb{Z}^n$ is the **n-torus**, and is denoted $\mathbb{T}^n$. This is an example of a **homogeneous space**.

(b.) Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via

$$(z^1, ..., z^n) = (x^1 + iy^1, ..., x^n + iy^n) \mapsto (x^1, y^1, ..., x^n, y^n)$$

we can view $S^{2n+1} \subset \mathbb{C}^n$ as

$$S^{2n+1} = \{(z^1, ..., z^n) \mid \sum |z^j|^2 = 1\}$$

Let m be a positive integer and $\ell_1, ..., \ell_n$ be positive integers relatively prime to m. Define an action of the cyclic group $\mathbb{Z}/m$ (the additive group of integers modulo m) on S^{2n+1} by

$$[k] \cdot (z^1, ..., z^n) = \left(e^{\frac{2k\pi i \ell_1}{m}} z^1, ..., e^{\frac{2k\pi i \ell_n}{m}} z^n\right)$$

for $[k] \in \mathbb{Z}/m$ (the residue of $k \in \mathbb{Z}$ modulo m). The quotient space is denoted $L_m(\ell_1, ..., \ell_n)$ and is called a **lens space**.
Hint: It might be useful in this exercise to think of S^{2n+1} as a smooth manifold in the sense of GP.

(c.) $GL_n(\mathbb{Z}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ by matrix multiplication $A \cdot B = AB$. This quotient is another example of a homogeneous space.

7. The set of complex lines in $\mathbb{C}^{n+1}$ is called **complex projective space** and is denoted $\mathbb{CP}^n$. It is the set of equivalence classes

$$\{[z^0 : .... : z^n] \mid (z^0, ..., z^n) \in \mathbb{C}^{n+1} \setminus \{0\}\}$$

where $[z^0 : ... : z^n] = [w^0 : .... : w^n]$ if and only if there exists $t \in \mathbb{C}^*$ so that $z^j = tw^j$ for each $j = 0, ..., n$.
Identifying $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ as in the previous problem, prove that $\mathbb{CP}^n$ can be made into a smooth (real) $2n$–manifold so that the projection

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$$

is a smooth map.

Hint: Define the same kinds of charts $\psi_j([z^0 : ... : z^n]) = (z^0, ..., \hat{z}^j, ..., z^n)$.