Math 520 Take-home final exam.

December 9, 2006

Please write (neatly) solutions to as many problems as you can (solving parts of problems is good, too!). These are due by 3:00pm on Tuesday, December 12.

In addition, please pick one problem to present to me during the final exam period, 8:00–11:00am, Tuesday, December 12. You will have 1/2-hour to present your solution, so please plan accordingly. I will email you on Monday with your 1/2-hour appointment time.

You can use your notes, books (GP and Boothby) and Prof. Lerman's notes only (no internet). You are to work on the exam on your own so please do not talk to others about the exam until after 3:00pm on Tuesday.

Good luck!!

All manifolds are assumed to be without boundary, unless otherwise stated. A closed manifold is a compact manifold (without boundary).

1. Let M, N be closed manifolds and $f : M \to N$ a surjective local diffeomorphism. Prove that for every $p \in N$, there exists $k \geq 1$ and a neighborhood U of p so that

$$f^{-1}(U) = \bigcup_{j=1}^{k} V_j$$

with $V_j \cap V_i = \emptyset$ for $i \neq j$ and $f|_{V_j} : V_j \to U$ is a diffeomorphism.

2. Let M be a connected (and so path connected) n–manifold, and $\alpha \in \Omega^1(M)$ a closed 1–form.
(a.) Prove that if $\phi : U \to \mathbb{B}_1^n$ is a local coordinate chart to the open 1–ball in $\mathbb{R}^n$, then $(\phi^{-1})^*(\alpha) = df$ for some $f : \mathbb{B}_1^n \to \mathbb{R}$ (equivalently, every closed 1–form on $\mathbb{B}_1^n$ is exact).
(b.) Assume now that α has **integral periods**—that is, suppose that for any (piecewise smooth) closed loop, $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1)$, we have

$$\int_{[0,1]} \gamma^*(\alpha) \in \mathbb{Z}$$

Prove that there is a smooth map

$$F : M \to S^1 = \mathbb{R}/\mathbb{Z}$$

so that $\alpha = F^*(\beta)$. Here, $\beta \in \Omega^1(S^1)$ is the unique 1–form with $\pi^*(\beta) = dt$, $\pi : \mathbb{R} \to S^1$ the quotient map and t the global coordinate on $\mathbb{R}$.

3. Let $f : U \to \mathbb{R}$ be a smooth function defined on an open set $U \subset \mathbb{R}^2$. The graph of f is a submanifold $graph(f) \subset \mathbb{R}^3$ parameterized by

$$\Phi : U \to graph(f)$$

with $\Phi(x^1, x^2) = (x^1, x^2, f(x^1, x^2))$. Giving $graph(f)$ the induced Riemannian metric from this embedding, compute the metric and Levi-Civita connection in the coordinates (x^1, x^2). That is, compute g_{ij} and Γ_{ij}^k with respect to the coordinates x^1, x^2.

4. Let Σ be a nonsingular symmetric $n \times n$ matrix. The associated quadratic form is given by

$$q(\mathbf{x}) = \mathbf{x}^T \Sigma \mathbf{x}$$

where $\mathbf{x}^T$ is the transpose of the n–dimensional column vector $\mathbf{x}$. The **orthogonal group of** q is the subgroup $\mathrm{O}(q) < \mathrm{GL}_n(\mathbb{R})$ defined by

$$\mathrm{O}(q) \quad = \quad \{A \,|\, q(A\mathbf{x}) = q(\mathbf{x})\} \quad = \quad \{A \,|\, A^T \Sigma A = \Sigma\}$$

Prove that $O(q)$ is a submanifold of $\mathrm{GL}_n(\mathbb{R})$.

5. Let M be a closed manifold, $\alpha \in \Omega^p(M)$ and $\xi \in \mathfrak{X}(M)$ a vector field with flow ϕ_t.
(a.) Prove that for all $s \in \mathbb{R}$, we have

$$L_\xi(\phi_s^*(\alpha)) = \phi_s^*(L_\xi \alpha)$$

(b.) Suppose that $L_\xi \alpha = 0$ and prove that for all $s \in \mathbb{R}$, we have

$$\frac{d}{dt}|_{t=s} \phi_t^*(\alpha) = 0$$

conclude that $\phi_t^*(\alpha) = \alpha$.
(c.) Now suppose that ω is a nowhere vanishing top-form on M (so M is oriented) and assume

$$L_\xi \omega = 0$$

Prove that for any $m \in M$, any neighborhood U of m, and any $T > 0$, there exists $t > T$ so that

$$\phi_t(U) \cap U \neq \emptyset$$

(*Hint: Note that for any nonempty open set $U \subset M$ we have*

$$0 < \int_U \omega \leq \int_M \omega < \infty$$

In particular, you may assume without proof that $\int_U \omega$ is well defined, even though $\omega|_U$ is not compactly supported.)