Differentiable Manifolds: Problem set 5

Due Monday October 6

Read GP §2.1–2.2. Do problems:

§2.1: 4, 5, 9;

1. Submanifolds $N, P \subset M$ are **transverse**, written $N \pitchfork P$ if for every $m \in N \cap P$
$$T_m N + T_m P = T_m M.$$ Equivalently, the embedding of $P$ into $M$ is transverse to $N$ (or equivalently, the embedding of $N$ into $M$ is transverse to $P$). Here embedding means injective immersion for which the inclusion is a homeomorphism onto its image.

Prove the following local transversality theorem:

**Theorem** If $N^k, P^r \subset M^n$ are embedded submanifolds and $N \pitchfork P$, then for every $m \in N \cap P$, there is a coordinate chart
$$\phi : U \to (-1, 1)^n$$ about $m$ in $M$, so that
$$\phi(N \cap U) = \{(a_1, \ldots, a_k, 0, \ldots, 0) \mid a_j \in (-1, 1)\}$$
$$\phi(P \cap U) = \{(0, \ldots, 0, a_{n-r+1}, \ldots, a_n) \mid a_j \in (-1, 1)\}$$

2. Let $U \subset \mathbb{R}^3$ and
$$\xi = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} - xz \frac{\partial}{\partial z} \text{ and } \eta = y^2 \frac{\partial}{\partial x} - xy \frac{\partial}{\partial y} - xz \frac{\partial}{\partial z}$$
compute $[\xi, \eta]$.

3. Let $M$ be a smooth manifold and $\xi, \eta, \zeta \in \mathfrak{X}(M)$, smooth vector fields on $M$. Verify the **Jacobi identity**:
$$[[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0$$
Along with the property $[\xi, \eta] = -[\eta, \xi]$, this shows that the (real) vector space $\mathfrak{X}(M)$ with the bilinear operation $[\cdot, \cdot]$ is a **Lie algebra**.
4. Recall that we have a canonical identification $T_g(\text{GL}_n(\mathbb{R})) = M_{n \times n}(\mathbb{R})$ for any $g \in \text{GL}_n(\mathbb{R})$. Given $A \in M_{n \times n}(\mathbb{R})$, we can construct a smooth vector field on $\text{GL}_n(\mathbb{R})$

$$\xi^A : \text{GL}_n(\mathbb{R}) \to T\text{GL}_n(\mathbb{R}) = \text{GL}_n(\mathbb{R}) \times M_{n \times n}(\mathbb{R})$$

by

$$\xi^A(g) = (g, gA)$$

We have global coordinates $x_{ij}$ on $M_{n \times n}(\mathbb{R})$, and hence also on $\text{GL}_n(\mathbb{R})$, with $x_{ij}(g) = g_{ij}$. We can thus express $\xi^A$ in terms of the associated basis of vector fields $\frac{\partial}{\partial x_{ij}}$ as

$$\xi^A = \sum_{i,j,k} x_{ik}A_{kj} \frac{\partial}{\partial x_{ij}}.$$ 

Therefore, $\xi^A$ is a smooth vector field on $\text{GL}_n(\mathbb{R})$.

Note that left multiplication by $g$

$$\ell_g : \text{GL}_n(\mathbb{R}) \to \text{GL}_n(\mathbb{R})$$

$$\ell_g(h) = gh$$

is actually the restriction to $\text{GL}_n(\mathbb{R})$ of a linear map $M_{n \times n}(\mathbb{R}) \to M_{n \times n}(\mathbb{R})$.

So the derivative map

$$d(\ell_g)_h : M_{n \times n}(\mathbb{R}) \to M_{n \times n}(\mathbb{R})$$

is just given by

$$d(\ell_g)_h(B) = gB$$

for any $h \in \text{GL}_n(\mathbb{R})$.

a. Check that

$$\beta : M_{n \times n}(\mathbb{R}) \to \mathfrak{X}(\text{GL}_n(\mathbb{R}))$$

given by

$$\beta(A) = \xi^A$$

is a linear injection. Thus the image of $\beta$ is a finite dimensional subspace of $\mathfrak{X}(\text{GL}_n(\mathbb{R}))$, which is denoted $\mathfrak{gl}_n(\mathbb{R})$.

b. Check that for every $g \in \text{GL}_n(\mathbb{R})$ the diffeomorphism $\ell_g$ preserves $\xi^A$ (that is, $\xi^A$ is $\ell_g$-related to itself).

c. Check that $\mathfrak{gl}_n(\mathbb{R})$ is closed under the Lie bracket, and in fact:

$$[\xi^A, \xi^B] = \xi^{AB-BA}$$

(it is a big computation, don’t be scared). This last matrix $AB-BA$ is often denoted $[A, B] = AB - BA$, so that the equation becomes

$$[\xi^A, \xi^B] = \xi^{[A,B]}$$

Thus, $\mathfrak{gl}_n(\mathbb{R})$ is a finite dimensional Lie subalgebra of $\mathfrak{X}(\text{GL}_n(\mathbb{R}))$ and is called the Lie algebra of $\text{GL}_n(\mathbb{R})$.  

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d. Check that if $A$ and $B$ are in $T_I(\text{SL}_n(\mathbb{R}))$, i.e. they have trace zero, then $[A, B] \in T_I(\text{SL}_n(\mathbb{R}))$. In particular, after restricting both domain and range, $\beta$ describes a linear injection of $T_I(\text{SL}_n(\mathbb{R}))$ into $\mathfrak{x}(\text{SL}_n(\mathbb{R}))$ and the image is also closed under Lie bracket. This image $\mathfrak{s}_n(\mathbb{R})$ is the Lie algebra of $\text{SL}_n(\mathbb{R})$.

e. Similarly, for the orthogonal group $\text{O}_n(\mathbb{R})$, check that if $A$ and $B$ are in $T_I(\text{O}_n(\mathbb{R}))$, i.e. $A + A^t = 0$ and $B + B^t = 0$ (see GP problem 10, §1.5), then $[A, B] \in T_I(\text{O}_n(\mathbb{R}))$. Again, after restricting appropriately, $\beta$ defines a linear injection of $T_I(\text{O}_n(\mathbb{R}))$ into $\mathfrak{x}(\text{O}_n(\mathbb{R}))$. The image $\mathfrak{o}_n(\mathbb{R})$ is the Lie algebra of $\text{O}_n(\mathbb{R})$.

Note that the Lie algebras $\mathfrak{g}_n(\mathbb{R})$ is isomorphic to $M_{n \times n}(\mathbb{R})$ with the bracket given by $[A, B] = AB - BA$. Similarly, $\mathfrak{s}_n(\mathbb{R})$ and $\mathfrak{o}_n(\mathbb{R})$ are isomorphic to Lie subalgebras of $M_{n \times n}(\mathbb{R})$. 