1. Consider the three points \(A = (0,1,0), B = (1,1,1),\) and \(C = (0,2,1)\) in \(\mathbb{R}^3\). For each part, circle the best answer. (1 point each)

(a) The projection of the vector \(\overrightarrow{AC}\) onto \(\overrightarrow{AB}\) is:

\[
\begin{pmatrix}
0, 2, 2 \\
\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \\
\frac{1}{2}, 0, \frac{1}{2} \\
\frac{1}{2}, \frac{1}{2}, 0
\end{pmatrix}
\]

(b) The area of the triangle formed by these three points is:

\[
\sqrt{3} \quad \frac{\sqrt{3}}{2} \quad 3 \quad 3 \quad \frac{3}{2}
\]

(c) \(\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = -3 - 2 - 1 0 1 2 3\)

(d) For the vector \(\mathbf{v} = (0, 2, 2)\), the angle between \(\overrightarrow{AC}\) and \(\mathbf{v}\) is:

\[
0 \quad \pi/4 \quad \pi/2 \quad \pi \quad 3\pi/2
\]

2. Suppose \(f: \mathbb{R}^2 \to \mathbb{R}\) has the table of values and partial derivatives shown at right. For \(x(r,s) = 2r - s\) and \(y(r,s) = s^2 - 4r\), let \(F(r,s) = f(x(r,s), y(r,s))\) be their composition with \(f\).

Circle the value of \(\frac{\partial F}{\partial r}(1,2)\):

\[
\begin{array}{cccc}
24 & -24 & 40 & -40 & -11 & 11 \\
\end{array}
\]

(2 points)

<table>
<thead>
<tr>
<th>(x,y)</th>
<th>f(x,y)</th>
<th>(\frac{\partial f}{\partial x})</th>
<th>(\frac{\partial f}{\partial y})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>3</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>(1,2)</td>
<td>6</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>(3,3)</td>
<td>19</td>
<td>-8</td>
<td>5</td>
</tr>
<tr>
<td>(4,3)</td>
<td>7</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

---

\[\text{Scratch Space}\]

1. a) \(\overrightarrow{b} = \overrightarrow{AB} = \langle 1, 0, 1 \rangle, \quad \overrightarrow{c} = \overrightarrow{AC} = \langle 0, 1, 1 \rangle\)

\[
\text{proj}_b \overrightarrow{c} = \frac{\overrightarrow{b} \cdot \overrightarrow{c}}{||\overrightarrow{b}||^2} \overrightarrow{b} = \frac{1}{2} \langle 1, 0, 1 \rangle = \langle \frac{1}{2}, 0, \frac{1}{2} \rangle
\]

b) \(\overrightarrow{b} \times \overrightarrow{c} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 0 & 1 \\
0 & 1 & 0
\end{vmatrix} = \langle -1, -1, 1 \rangle, \text{ so area} = \frac{1}{2} ||\overrightarrow{b} \times \overrightarrow{c}|| = \sqrt{3}/2.

c) \(\overrightarrow{b}\) is \(\perp\) to \(\overrightarrow{b} \times (\text{anything})\).

d) \(\overrightarrow{b}\) and \(\overrightarrow{c}\) point in the same direction.

2. \(\frac{\partial F}{\partial r}(1,2) = \frac{\partial F}{\partial x}(x(1,2), y(1,2)) \cdot \frac{\partial x}{\partial r}(1,2) + \frac{\partial F}{\partial y}(x(1,2), y(1,2)) \cdot \frac{\partial y}{\partial r}(1,2)\)

\[
= 4 \cdot 2 + 8 \cdot (-4) = -24.
\]
3. Exactly one of the limits at right exists. Circle the limit that exists:

(A) \[ \lim_{(x,y) \to (0,0)} \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} \]

(B) \[ \lim_{(x,y) \to (0,0)} \frac{x^2 + y}{\sqrt{x^2 + y^2}} \]

Justify your answer either by showing that the limit you circled exists and computing its value, or by showing that the other limit does not exist. (3 points)

(A) exists: In polar, the fn is \( \frac{\sin r}{r} \) and so the limit is equivalent to the 1-var limit \( \lim_{r \to 0} \frac{\sin r}{r} \)

\[ = \lim_{r \to 0} \frac{\cos r}{1} = 1 \text{ by L'Hopital's Rule.} \]

(B) Does not exist: Again in polar, the fn is \( \frac{r \cos^2 \theta + \cos \theta}{r} \).

Thus, if we approach along a line \( (\text{so } \theta = \text{fixed}) \) we get \( \cos \theta \) as the limit, which can be anything in \([-1, 1]\). So the limit D.N.E.

4. When \( x, y > 0 \), \( f = x + y \) which is differentiable at \((1,1)\). At \((0,0)\), \( f \) doesn't jump, but there isn't a good tangent plane.

4. Consider the function \( f(x, y) = |x| + |y| \), whose graph is shown at right. Circle the phrase that best completes each sentence. (1 point each)

(a) At the point \((1, 1)\), the function \( f \) is \[ \text{continuous} \quad \text{differentiable} \quad \boxed{\text{both continuous and differentiable}} \]

(b) At the point \((0, 0)\), the function \( f \) is \[ \text{continuous} \quad \boxed{\text{differentiable}} \quad \text{both continuous and differentiable} \]
5. Suppose that \( g(x, y, z) \) is a function and \( g(1, 0, 3) = 6 \) and \( \nabla g(1, 0, 3) = 2i + 3j + 4k \). Use linear approximation to estimate the value of \( g(1.2, -0.1, 2.9) \). (2 points)

\[
g(1.2, -0.1, 2.9) \approx \begin{bmatrix} 5.5 & 5.7 & 5.8 & 5.9 & 6 & 6.1 & 6.2 & 6.3 & 6.4 & 6.5 \end{bmatrix}
\]

6. The level curves of the partial derivatives \( f_x \) (solid lines) and \( f_y \) (dashed lines) of a function \( f(x, y) \) are shown at right. There are exactly two critical points of \( f \) in the domain shown in the picture. Find both of them and classify each as a local minimum, local maximum, or saddle. (3 points)

Critical pts are where \( f_x = f_y = 0 \), i.e. where the level sets \( f_x = 0 \) and \( f_y = 0 \) cross.

(0, 0) which is a local min

(2, 2) which is a local min

Scratch Space

\[
5. \quad g(1.2, -0.1, 2.9) \approx g(1, 0, 3) + g_x(1, 0, 3) \cdot (0.2) + g_y(-) \cdot (-0.1)
\]

\[
+ g_z(-) \cdot (-0.1)
\]

\[
= 6 + 2 \cdot 0.2 + 3 \cdot (-0.1) + 4 \cdot (-0.1) = 5.7
\]

At \((0, 0)\), have \( f_{xx} > 0 \), \( f_{yx} = 0 \), and \( f_{yy} > 0 \). Hence \( D = \left|\begin{array}{cc} f_{xx} & f_{yx} \\ f_{yx} & f_{yy} \end{array}\right| > 0 \) and \( f_{xx} > 0 \Rightarrow \text{local min} \)

At \((2, 2)\), have \( f_{xx} = 0 \), \( f_{yy} = 0 \) and \( f_{xy} < 0 \) so \( D < 0 \Rightarrow \text{saddle} \).
7. Find the absolute maximum and minimum values of the function \( f(x, y) = 2x^2 + y^2 + 5 \) subject to the constraint \( x^2 + y^2 \leq 4 \). (3 points)

Lagrange Multi: \( \nabla f = \langle 4x, 2y \rangle = \lambda \nabla g = \lambda \langle 2x, 2y \rangle \)

Eqns: \( 2x = \lambda x \quad y = \lambda y \quad x^2 + y^2 = 4 \)

\( x \neq 0 \Rightarrow \lambda = 2 \Rightarrow y = 2y \Rightarrow y = 0 \Rightarrow x = \pm 2 \)
\( y \neq 0 \Rightarrow \lambda = 1 \Rightarrow 2x = x \Rightarrow x = 0 \Rightarrow y = \pm 2 \).

So four crit pts: \( (\pm 2, 0) \) where \( f = 13 \)
\( (0, \pm 2) \) where \( f = 9 \)

Interior Crit Pts: \( \nabla f = \langle 4x, 2y \rangle = 0 \), i.e.
\( x = y = 0 \) and \( f = 5 \).

The Extreme Value Thm says that since \( f \) is cont. and \( D = \{x^2 + y^2 \leq 4\} \) both closed and bound then \( f \) has abs min/max on \( D \) which must occur at the five pts enumerated above.

Hence:

\[
\begin{array}{l}
\text{Absolute max value} = 13 \\
\text{Absolute min value} = 5
\end{array}
\]
8. The contour map of a differentiable function $f$ is shown at right. For each part, circle the best answer. (2 points each)

(a) The directional derivative $D_uf(P)$ is:
- positive
- negative
- zero

(b) Estimate $\int_C f \, ds$:
- 8.1 
- 5.4 
- 2.7 
- 0
- 2.7
- 5.4
- 8.1

(c) Estimate $\iint_R f \, dA$:
- 4.8 
- 3.2 
- 1.6 
- 0
- 1.6
- 3.2
- 4.8

(d) The point $Q$ is:
- a local maximum
- a local minimum
- a saddle
- not a critical point

(e) Find $\int_C \nabla f \cdot dr$:
- 12
- 9
- 6
- 3
- 0
- 3
- 6
- 9
- 12

---

Scratch Space

b) $\int_C f \, ds = (\text{Average of } f \text{ on } C) (\text{length of } C)$
\[ \approx (4 \text{ since min on } C \text{ is } 0) (1.2 \text{ since line joining endpoints has length } 1) \]
= 4.8

c) $\iint_R f \, dA = (\text{Average of } f \text{ on } C) (\text{Area } R)$
\[ \approx (6) (1/2 \times 1/2) = 6 \times 1/4 = 3/2 = 1.5 \]

d) $f$ increases as we move horizontally and decreases as we move vertically.

e) $\int_C \nabla f \cdot dr = f(B) - f(A) = 0 - 0 = 0$
by the Fund. Thm of Line Integrals.
9. Suppose that \( r : [0, 2] \to \mathbb{R}^2 \) is a parametric curve in the plane and that \( r(t) \) and \( r'(t) \) have the values given in the table at left. Check the box below to the picture that could be a plot of \( r(t) \). (2 points)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( r(t) )</th>
<th>( r'(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0, 0)</td>
<td>i</td>
</tr>
<tr>
<td>1</td>
<td>(1, 1)</td>
<td>-i</td>
</tr>
<tr>
<td>2</td>
<td>(2, 4)</td>
<td>i + 4j</td>
</tr>
</tbody>
</table>

10. For each of the integrals below, label the picture of the corresponding region of integration. (2 points each)

(A) \( \int_{1}^{2} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\pi} f(\rho, \phi, \theta) \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho \)

(B) \( \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\phi} g(\rho, \phi, \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \)
11. (a) Consider the vector field \( \mathbf{F} = (yz, -xz, yx) \) on \( \mathbb{R}^3 \). Compute the curl of \( \mathbf{F} \). (2 points)

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} y & \frac{\partial}{\partial y} x & \frac{\partial}{\partial z} x \\
yz & -xz & yx
\end{vmatrix} = \left( x + x \right) \hat{i} - \left( y - y \right) \hat{j} + \left( -z - z \right) \hat{k}
\]

\[\text{curl} \mathbf{F} = \langle 2x, 0, -2z \rangle\]

(b) Let \( S \) be the portion of the sphere \( x^2 + y^2 + z^2 = 13 \) where \( x \leq 3 \). Find the flux of curl \( \mathbf{F} \) through \( S \) with respect to the outward pointing unit normal vector field. (5 points)

By Stokes', \( \iint_S (\text{curl} \mathbf{F}) \cdot \mathbf{n} \, dA = \oint_C \mathbf{F} \cdot d\mathbf{r} \). Here \( C \) is the circle in the \( x=3 \) plane of radius 2 since \( 3^2 + y^2 + z^2 = 13 \Rightarrow y^2 + z^2 = 4 \), oriented as shown. So can use \( \mathbf{r}(t) = \langle 3, 2 \sin t, 2 \cos t \rangle \) for \( 0 \leq t \leq 2\pi \) to param. Now \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_0^{2\pi} \langle 2y, -2z, 2x \rangle \cdot \langle 0, 2 \cos t, -2 \sin t \rangle \, dt = \int_0^{2\pi} 4 \sin t \cos t - 4 \sin^2 t \cos t \, dt = \int_0^{2\pi} -12 \cos^2 t + 12 \sin^2 t \, dt = \int_0^{2\pi} -24 \, dt = -24\pi\]

\[\text{flux} = -24\pi\]

(c) Let \( D \) be the portion of sphere \( x^2 + y^2 + z^2 = 13 \) where \( x \geq 3 \). Find the flux of curl \( \mathbf{F} \) through \( D \) with respect to the outward pointing unit normal vector field. (2 points)

For \( D \), get the other orient on \( C \), so ans:

\[\text{flux} = 24\pi\]
12. (a) Let $S$ be the portion of the surface $z = -\sin(x)\sin(y)$ where $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$ which is shown at right. Use a parameterization to find the flux of $\mathbf{F} = (0, 0, 2z + 1)$ through $S$ with respect to the downward normals. (3 points)

\[
\vec{r}(u,v) = \langle u, v, -\sin u \sin v \rangle \quad 0 \leq u, v \leq \pi.
\]

\[
\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ u & v & 0 \\ 0 & 1 & -\sin u \cos v \end{vmatrix} = \langle \cos u \sin v, \sin u \cos v, 1 \rangle
\]

(upwards, so use $-\vec{r}_u \times \vec{r}_v$)

\[
\iint_S \mathbf{F} \cdot \vec{n} \, dA = \iiint_E \mathbf{F} \cdot \vec{D} \, dV
\]

\[
= \int_0^\pi \int_0^\pi 2 \sin u \sin v - 1 \, du \, dv
\]

\[
= 2 \int_0^\pi \sin u \, du \int_0^\pi \sin v \, dv - \int_0^\pi \int_0^\pi 1 \, du \, dv = 2 \cdot 2 \cdot 2 - \pi^2
\]

\[
= -\cos u \bigg|_{u=0}^{u=\pi} = 2
\]

\[
\text{flux} = 8 - \pi^2
\]

(b) Let $E$ be the region below the $xy$-plane and above $S$. Use an integral theorem to compute the flux of $\mathbf{F}$ through $\partial E$ with respect to the outward normals. (3 points)

\[
\text{Divergence Theorem: } \iiint_E \text{div } \mathbf{F} \, dV = \iint_{\partial E} \mathbf{F} \cdot \vec{n} \, dA
\]

\[
= \int_0^\pi \int_0^\pi 2 \, dz \, dy \, dx = \int_0^\pi \int_0^\pi 2 \sin x \sin y \, dy \, dx
\]

\[
= 2 \int_0^\pi \sin x \, dx \int_0^\pi \sin y \, dy = 8
\]

\[
\text{flux} = 8
\]

(c) Your answers in (b) and (c) should differ. Explain what accounts for the difference. (1 point)

$\partial E$ consists of both $S$ and the square $R = \{0 \leq x, y \leq \pi\}$ in the $xy$-plane. The flux through $R$ is

\[
\iint_R \mathbf{F} \cdot \vec{n} \, dA = \iint_R \langle 0, 0, 1 \rangle \cdot \langle 0, 0, 1 \rangle \, dA = \text{Area}(R) = \pi^2
\]
13. (a) Let \( S \) be the surface parameterized by \( \mathbf{r}(u, v) = (v \cos u, v, v \sin u) \) for \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 2 \). Mark the box next to its picture. 

(b) Use the parameterization to find the tangent plane to \( S \) at \((1, \sqrt{2}, 1)\). 

This pt has \( v = y = \sqrt{2} \) and \( x = 1 = \sqrt{2} \cos u \Rightarrow u = \pi/4 \). Now \( \mathbf{r}_u \times \mathbf{r}_v \) is \[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-v \sin u & 0 & v \cos u \\
\cos u & 1 & -v \sin u
\end{vmatrix} = \langle -v \cos u, v, -v \sin u \rangle
\]
which is \( \langle -1, \sqrt{2}, -1 \rangle \) when \( u = \pi/4 \) and \( v = \sqrt{2} \). Hence the tangent plane is
\[-1(x-1) + \sqrt{2}(y-\sqrt{2}) - 1(z-1) = 0 \quad \text{or} \quad \text{Equation: } -1 + \sqrt{2}y - z = 0\]

c) The surface \( S \) has area \( 4\sqrt{2}\pi \). Find the average of \( f(x, y, z) = y \) on \( S \). 

\[
\int_S f \, dA = \int_0^2 \int_0^{2\pi} \sqrt{\mathbf{r}_u \times \mathbf{r}_v} \, du \, dv = \int_0^{2\pi} \int_0^2 \sqrt{2} \, dv \, du
\]
\[
f(\mathbf{r}(u,v))
\]
\[
= \int_0^{2\pi} \int_0^2 \frac{\sqrt{2}}{3} \, dv \, du = \int_0^{2\pi} \frac{8\sqrt{2}}{3} \, du = \frac{16\sqrt{2} \pi}{3}
\]
So
\[
\text{Average } = \frac{16\sqrt{2} \pi / 3}{4\sqrt{2} \pi} = \frac{4}{3}
\]
14. Here are plots of six vector fields on the box where \(0 \leq x \leq 1\), \(0 \leq y \leq 1\), and \(0 \leq z \leq 1\). For each part, circle the best answer. (1 point each)

(a) The vector field given by \((z, 1, 0)\) is: \[ \text{A B C D E F} \]

(b) Exactly one of these vector fields has nonzero divergence. It is: \[ \text{A B C D E F} \]

For this example, the divergence is generally: \[ \text{negative \hspace{1cm} positive} \]

(c) The vector field \(A\) is conservative: \[ \text{true} \hspace{1cm} \text{false} \]

(d) Exactly one of the vector fields is constant, that is, independent of position. It is: \[ \text{A B C D E F} \]

(e) The vector field \(\text{curl C}\) is constant. The value of \(\text{curl C}\) is:

\[
\begin{bmatrix}
1 & -1 & j & -j & k & -k & 0
\end{bmatrix}
\]

(f) The vector field that is the gradient of a function \(f\) whose level sets are shown at right is: \[ \text{A B C D E F} \]
15. Let $S$ be the ellipsoid \( \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1 \) which is shown at right. Give a parameterization \( r: D \to \mathbb{R}^3 \) for \( S \), being sure to specify the domain \( D \) of the parameterization in the \((u,v)\)-plane. (3 points)

The transformation \((x, y, z) \mapsto (2x, y, 3z)\) takes the unit sphere to this ellipsoid.

Combining w/ our spherical coor, we can use

\[
D = \left\{ 0 \leq u \leq \pi , \ 0 \leq v \leq 2\pi \right\}
\]

\[
r(u, v) = \langle 2 \sin u \cos v, \sin u \sin v, 3 \cos u \rangle
\]

16. For each transformation \( R^2 \to R^2 \) below, circle “yes” or “no” depending on whether or not it takes the rectangle \( 0 \leq u \leq 1, \ 0 \leq v \leq 2 \) in the \((u,v)\)-plane to the parallelogram in the \((x,y)\)-plane with vertices \((0,0),(4,2),(2,-2),\) and \((6,0)\). (1 point each)

\[
a) \ \text{no} \quad T(u, v) = (2u + 4v, -2u + 2v) \\
b) \ \text{no} \quad T(u, v) = (4u + v, 2u - v) \\
c) \ \text{no} \quad T(u, v) = (4u + 2v, 2u - 2v) \\
d) \ \text{yes} \quad T(u, v) = (2u + 2v, -2u + v)
\]

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**Scratch Space**

Any correct \( T \) must take \((1,0)\) to either \((2,-2)\) or \((4,2)\) and take \((0,2)\) to the other one.

\[
a) \ T(1,0) = (2, -2) \quad T(0,2) = (8, 4) \\
b) \ T(1,0) = (4, 2) \quad T(0,2) = (2, -2) \checkmark \\
c) \ T(1,0) = (4, 2) \quad T(0,2) = (4, -4) \\
d) \ T(1,0) = (2, -2) \quad T(0,2) = (4, 2) \checkmark
\]
17. Let \( \mathbf{F} = (-x^3 \cos^3(x) + y^3, y^3 \sin^3(y) - x^3) \) and let \( C \) be the unit circle in the plane \( \mathbb{R}^2 \) oriented counterclockwise. Determine whether the integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is: negative \( \boxed{\text{negative}} \) zero positive \( \boxed{\text{positive}} \) (2 points)

18. Let \( S \) and \( H \) be the surfaces at right; the boundary of \( S \) is the unit circle in the \( xy \)-plane, and \( H \) has no boundary. Suppose there is a positive charge \( Q \) placed at the origin and let \( \mathbf{E} \) be the resulting electrical field. For each part, circle the correct answer. (2 points each)

(a) The flux \( \iint_H \mathbf{E} \cdot \mathbf{m} \, dS \) is: negative \( \boxed{\text{negative}} \) zero positive \( \boxed{\text{positive}} \)

(b) The flux \( \iint_S \mathbf{E} \cdot \mathbf{n} \, dS \) is: negative \( \boxed{\text{negative}} \) zero positive \( \boxed{\text{positive}} \)

(c) For \( \mathbf{G} = (xy^2, yz, x+z) \), the flux \( \iint_H (\operatorname{curl} \mathbf{G}) \cdot \mathbf{m} \, dS \) is: negative \( \boxed{\text{negative}} \) zero positive \( \boxed{\text{positive}} \)

---

**Scratch Space**

17. By Mr. Green: \( \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\text{Disc}} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \)

\[
= \iint_{\text{Disc}} -3x^2 - 3y^2 \, dA < 0 \text{ since the integrand is } < 0 \text{ (except at the origin).}
\]

18a. \( H \) does not contain the charge \( Q \), so this is 0 by Gauss's Law.

18b. As \( \operatorname{div} \mathbf{E} = 0 \), flux through \( S \) is the same as that of the lower hemisphere with the same boundary. The field \( \mathbf{E} \) is normal to \( L \) everywhere, and so flux is > 0.

18c. Since \( H \) is closed, \( \iint_H (\operatorname{curl} \mathbf{G}) \cdot \mathbf{m} \, dS = 0 \).